Center symmetry, neutral bions and (a tiny little bit of) Resurgence theory

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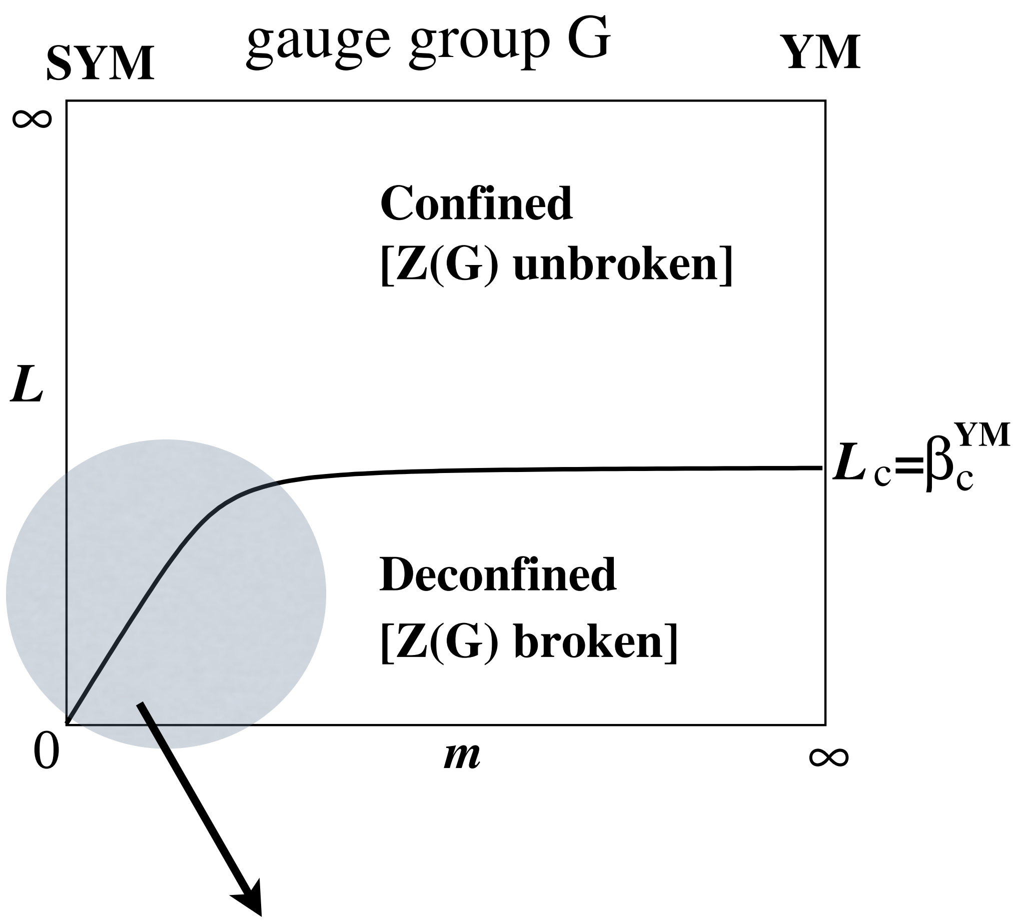
In collaboration with
Thomas Schaefer, Erich Poppitz

related works (on resurgence) with
Gerald Dunne, Philip Argyres,
Aleksey Cherman, Daniele Dorigoni, Gokce Basar.

MAIN RESULT

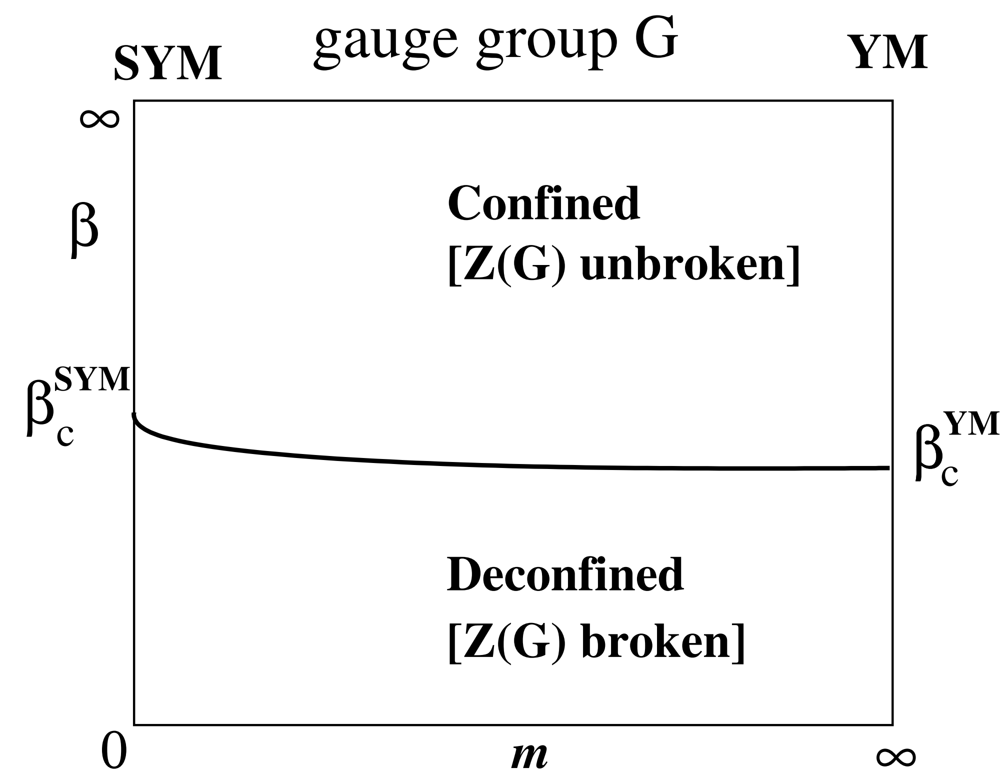
Yang-Mills theory with gauge group G and one Weyl fermion mass m, phase diagram in L-m plane

Spatial compactification



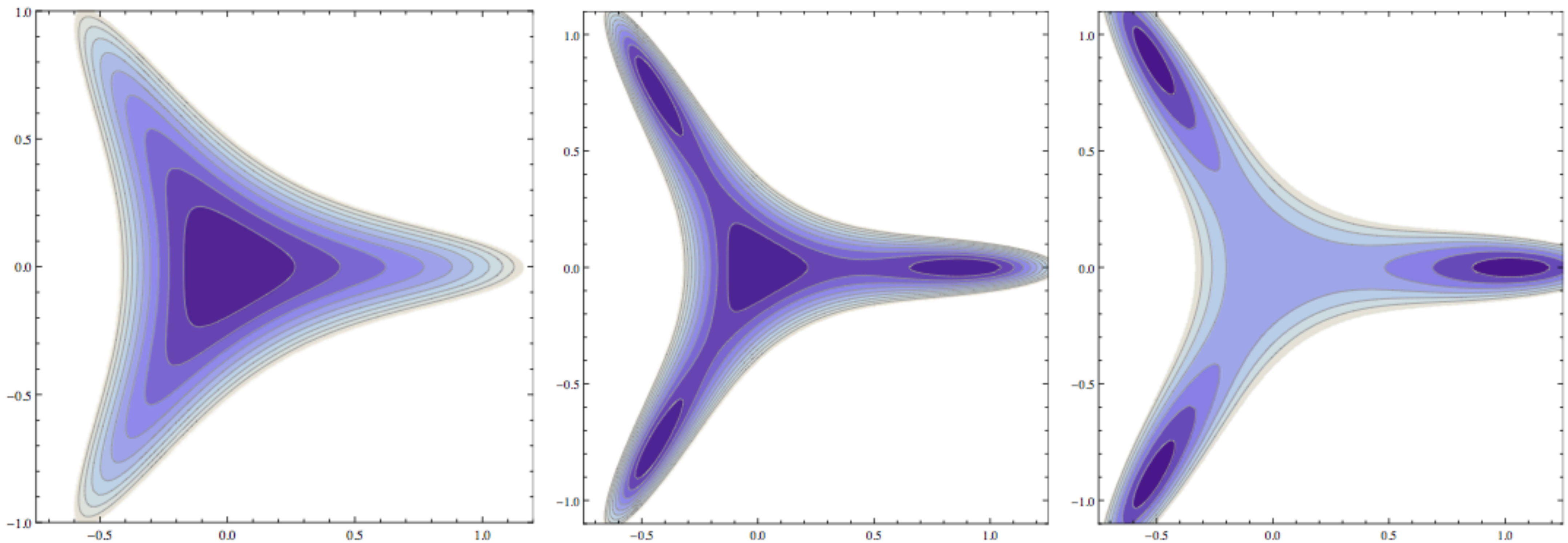
NP-Calculable,
in g^2 and $\exp[-A/g^2]$.
NP-effects
under control !

Thermal compactification



SU(3): Extrama of the NP-potential for trace of Wilson line, large-L to small-L.

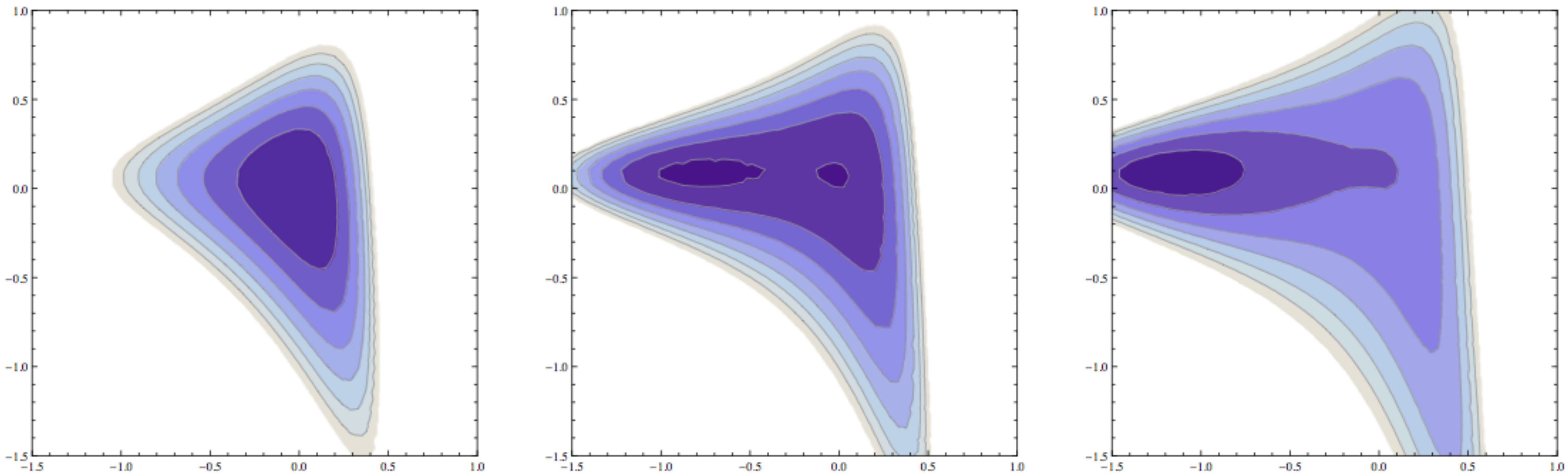
Not a model; no tuned parameters, result of justified semi-classics.



Confined | between the limits of metastability | Deconfined

G₂: Extrama of the NP-potential for trace of Wilson line, large-L to small-L

G_2: First order transition without change of symmetry.



Both qualitatively and quantitavely (numerical ratio of jump) identical to LGT result for YM.

LGT
[1] K. Holland, P. Minkowski, M. Pepe and U. J. Wiese,
[2] M. Pepe and U. -J. Wiese,
[3] J. Greensite, K. Langfeld, S. Olejnik, H. Reinhardt and T. Tok,
[4] G. Cossu, M. D'Elia, A. Di Giacomo, B. Lucini and C. Pica,

A non-historial, but hopefully coherent background to these results on calculable phase transition and confinement.

I will give a more general background than what the aforementioned results require.

The reason I do so is because the more general framework gives an opportunity to give a NP-definition of QFT in continuum.

Since I think this audience has an appreciation for both, I will try to do my best to explain in broad terms the underlying ideas from physics and mathematics.

Motivation: Can we make sense out of QFT? When is there a continuum definition of QFT?

Dyson(50s), 't Hooft (77),

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral (as we will). There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (see e.g. K. Costello's work).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

Recently, few people are attempting to answer this question, whether/when a NP continuum definition of QFT may exist and reinvigorate this problem.

Argyres, Dunne, MÜ: Resurgence in QFTs, QM, and path integrals
Schiappa, Marino,Aniceto..: Resurgence in string theory and matrix models
Kontsevich: recent talk at PI, Resurgence from the path integral perspective
Garoufalidis, Costin: Math and Topological QFTs

The common concept, which all these folks seem to be highly influenced by (and which is virtually unknown in physics community) is a "recent" mathematical progress, called

Resurgence Theory, developed by Jean Ecalle (80s)

and applied to QM by Pham, Delabaere, Voros.
(also relevant Dingle-Berry-Howls)

Ecalle's theory changed (will change?) the overall perspective on asymptotic analysis, for both mathematicians and physicists alike.

There are earlier hints that a resurgent structure must underlie QFT.

YM on R4 and standard problems

verbatim in 2d CP(N)

An asymptotically free QFT.

1) Perturbation theory is an asymptotic *(divergent) expansion even after regularization and renormalization*. Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R4. DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption! (Sadly, semi-classics is an awfully abused concept in literature!)

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined phase. (GPY, 80, ..) So, *no semi-classical approximation for the confined phase* up until recently is found, (except a supersymmetric version of the theory, due to reasons not related to supersymmetry, however, people thought it was due to SUSY!).

5) Incompatibility of large-N results with instantons. Obvious, must be so.

6) The IR-renormalon ambiguity (deeper, to be explained), ('t Hooft,79).

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence literature

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

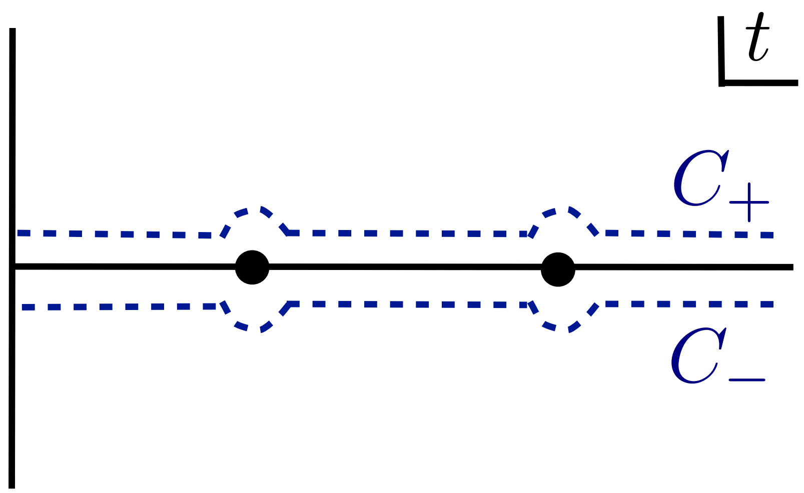
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt\ .$$

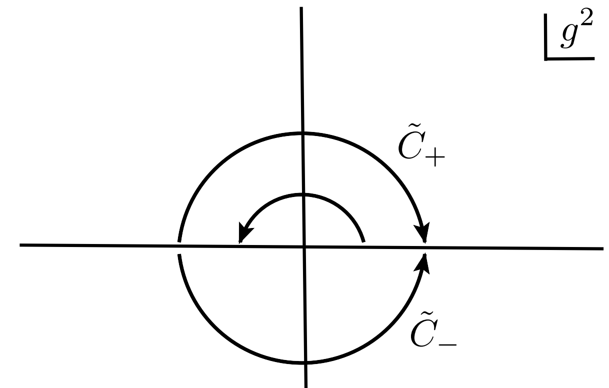
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Borel plane and lateral (left/right) Borel sums

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



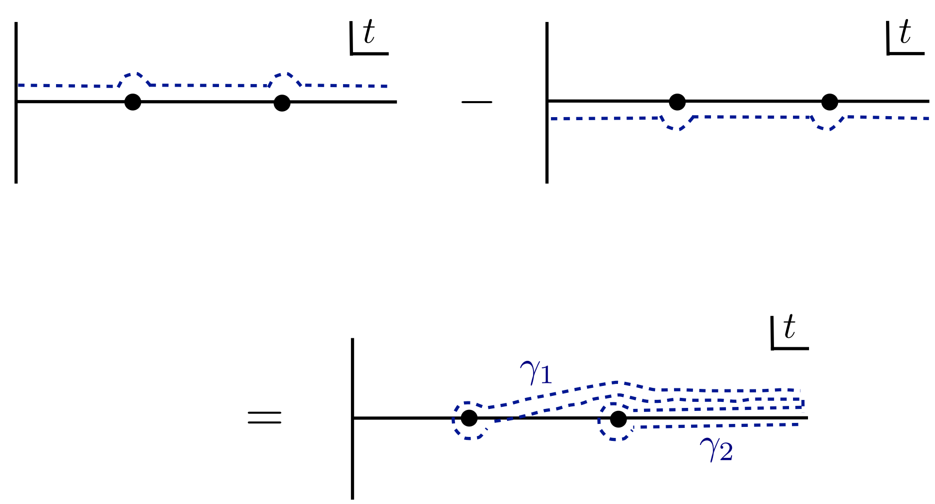
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0^\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). This happens because we are on a Stokes line (1850's)



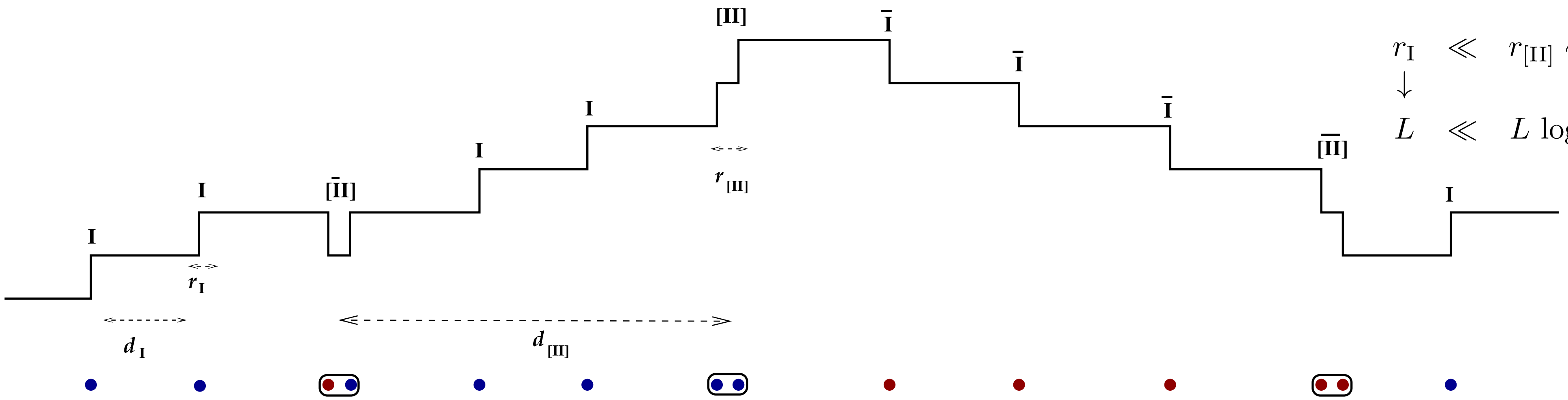
$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta^-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta^-}),$$

$$\mathrm{Disc}_{\theta^-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \dots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Jean Ecalle, 80s

Bogomolny--Zinn-Justin (BZJ) prescription

Bogomolny-Zinn-Justin prescription in QM (80s): done for double well potential, but consider a periodic potential. Dilute instanton, molecular instanton gas.

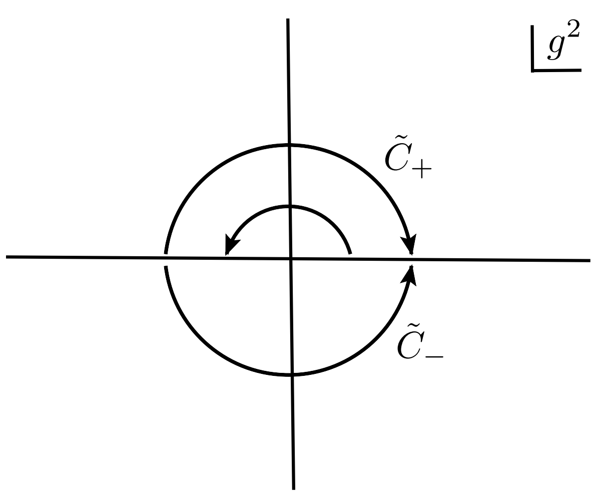


$$\begin{array}{ccccccc} r_{\rm I} & \ll & r_{[\rm II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{[\rm II]}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L & \ll & L\,\log\left(\frac{1}{g^2}\right) & \ll & L\,e^{S_0} & \ll & L\,e^{2S_0}. \end{array}$$

How to make sense of topological molecules (or molecular instantons)? Why do we even need a molecular instanton? (Balitsky-Yung in SUSY QM, (86))

Naive calculation of I-anti-I amplitude: meaningless (why?) at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate the integral, and continue back to $g^2 > 0$: two fold-ambiguous!

$$[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\overline{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm}$$

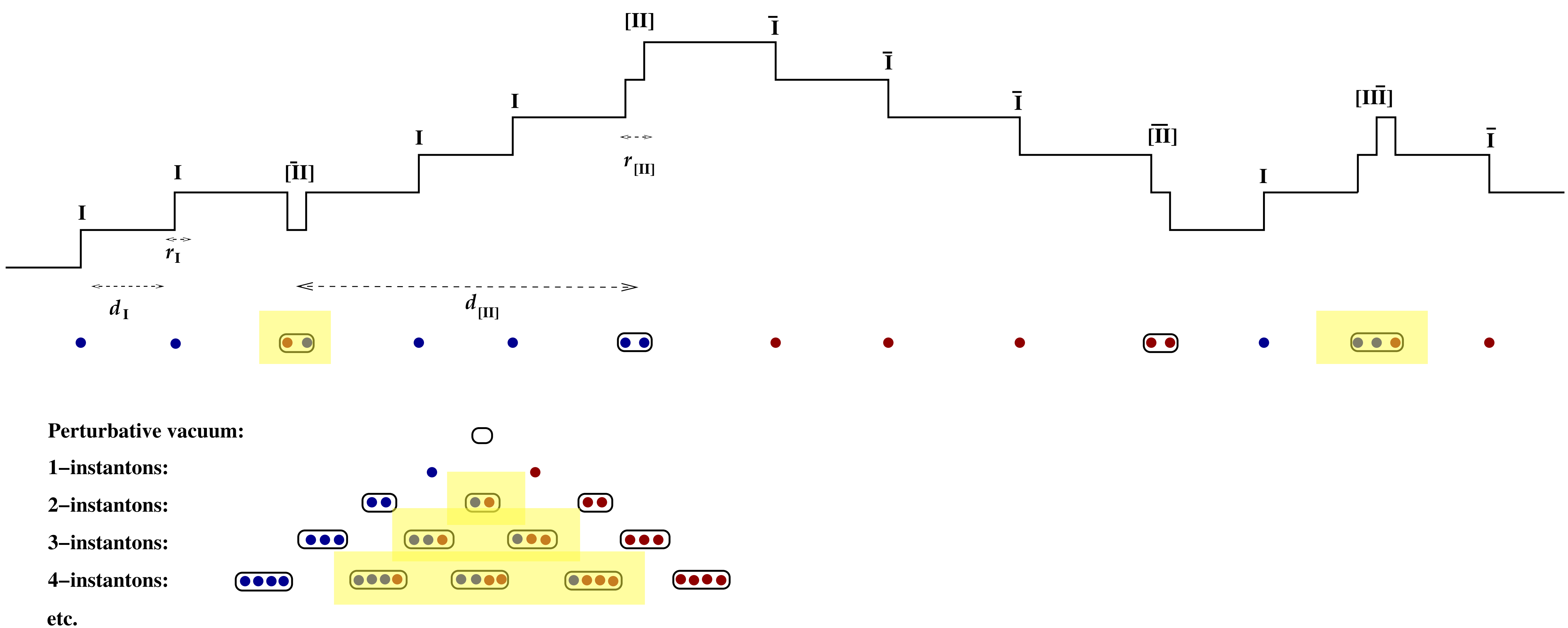


Why?: because we are (as before) on Stokes line!

Remarkable fact: Leading ambiguities cancel. "N.P. CONFLUENCE EQUATION", elementary incidence of Borel-Ecalle summability which I will return:

$$\mathrm{Im}\, \mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\, [\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

The topological configurations with multifold ambiguous amplitudes:
All are non-BPS quasi-solutions!



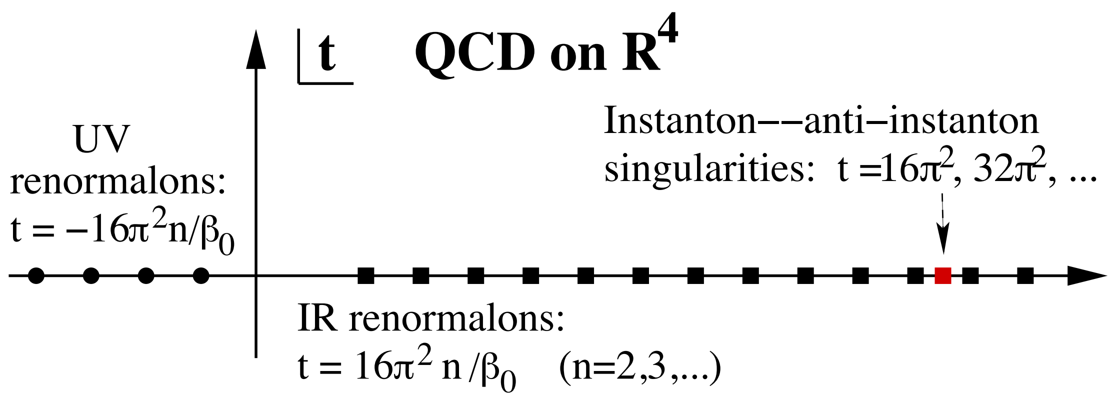
Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :***No,*** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : ***No,*** *on R2.* Dunne, MÜ: ***Yes***, *on R1 x S1*

Why doesn't it work? on R4 or R2?

Instanton-anti-instanton contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2 S_I$. (Modulo the standard IR problems with 2d instantons, also see Bogomolny-Fateyev(77)).

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams!)



't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons.
No such configuration is known!!

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

Can this work in QFT? QCD on R4 or CP(N-1) on R2?

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Standard view in late 70s, from Parisi(78)

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8] .

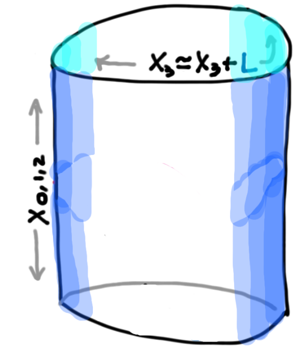
5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

YM/QCD on R3 x S1:
Idea of adiabatic continuity



Phase transition or
rapid cross-over

high $-\ T$ low $-\ T$

$\mathbb{R}^{d-1}$ $\mathbb{R}^{d-1} \times \mathbb{S}^1_\beta$ $\mathbb{R}^d$

$$V_{1-\text{loop}}[\Omega] = (-1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\,|\text{tr}\,\Omega^n|^2 \qquad \Omega = e^{\int_{S^1} A}$$

Gross, Pisarski, Yaffe 1980

We want continuity:

$\mathbb{R}^{d-1} \times \mathbb{S}^1_L$

Prevent phase-transition by using circle compactification
(pbc for fermions) or double-trace deformation!

One-loop potential for QCD(adj) on R3 x S1

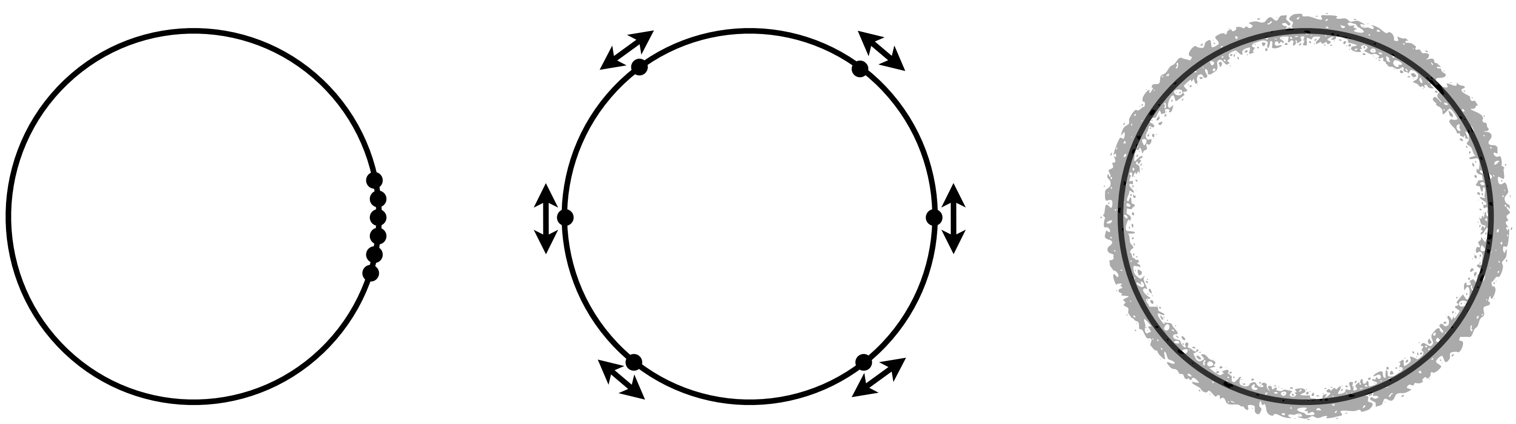
$$V_{1-\text{loop}}[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left(-1 + N_f\right) \left|\text{tr}\, \Omega^n\right|^2$$

$-1 < 0$ instability, “calculations between 1980-2007”
(modulo Hosotani (88), but this was unknown in QCD community, at least to us.)

$-1 + 1 = 0$ Supersymmetric case, $N_f = 1$, marginal, NP-stable

$-1 + 2 = 1 > 0$ QCD(adj), $N_f > 1$, **stability.**

Kovtun, MU, Yaffe,07,
Hosotani 88.



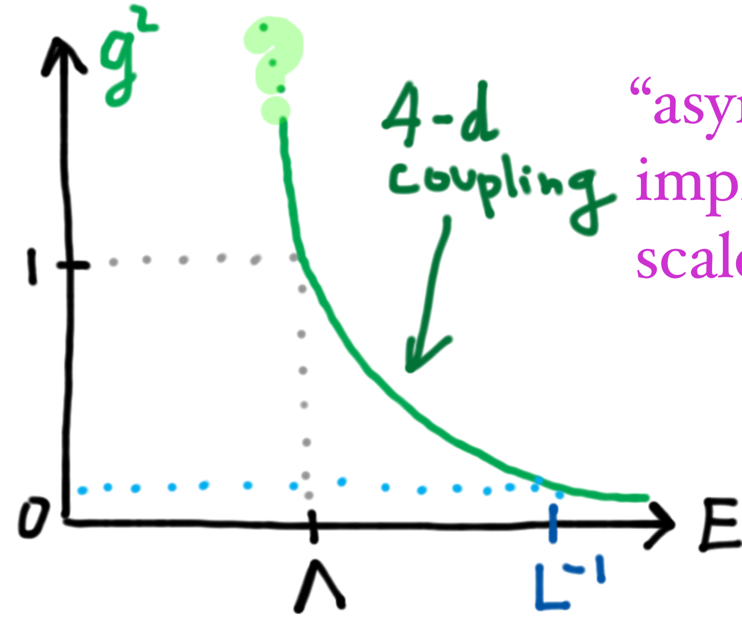
(a) (b) (c)

$NL\Lambda < 2\pi$
Abelian confinement,
volume dependence.

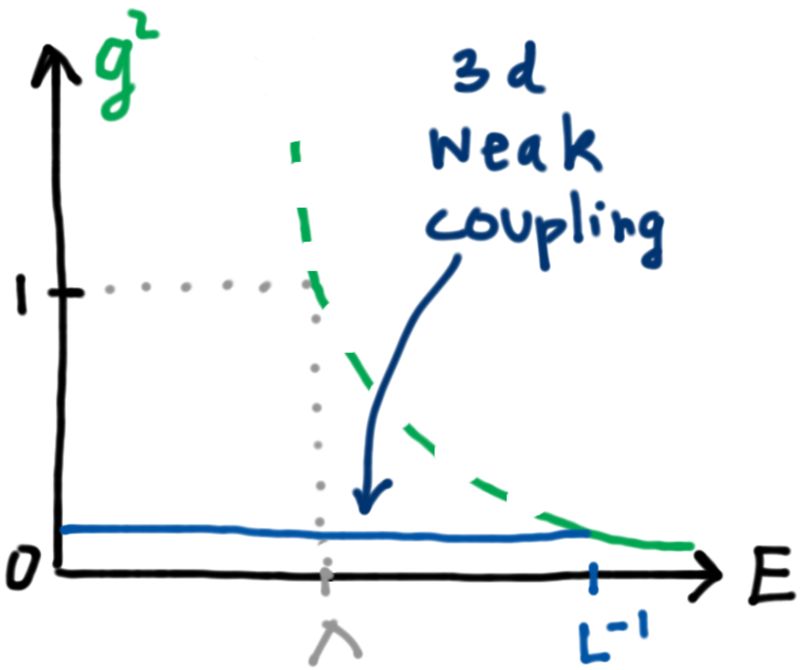
$NL\Lambda >> 2\pi$
Non-abelian confinement,
volume independence.

(a)Weak coupling trivial hol.
(b) Weak-coupling non-trivial hol.
(c) Strong-coupling non-trivial hol.

Scales and running coupling! Be careful of crucial difference wrt to thermal case.



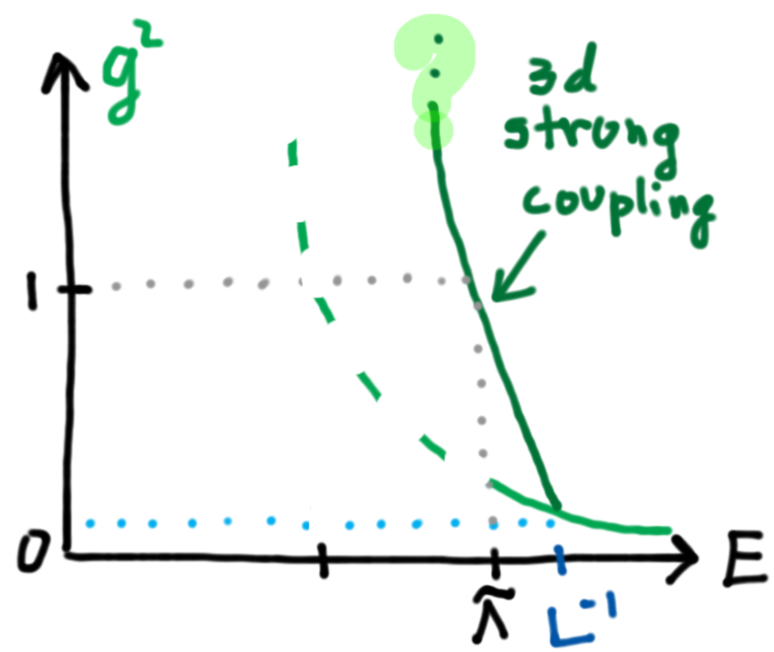
"asymptotic freedom" implies weak coupling at the scale of compactification!



spatial circle
deformed YM
QCD(adj) pbc

Yaffe, MÜ 08, Argyres, MÜ, 12,
Ogilvie, Myers, Meisinger 07/08...

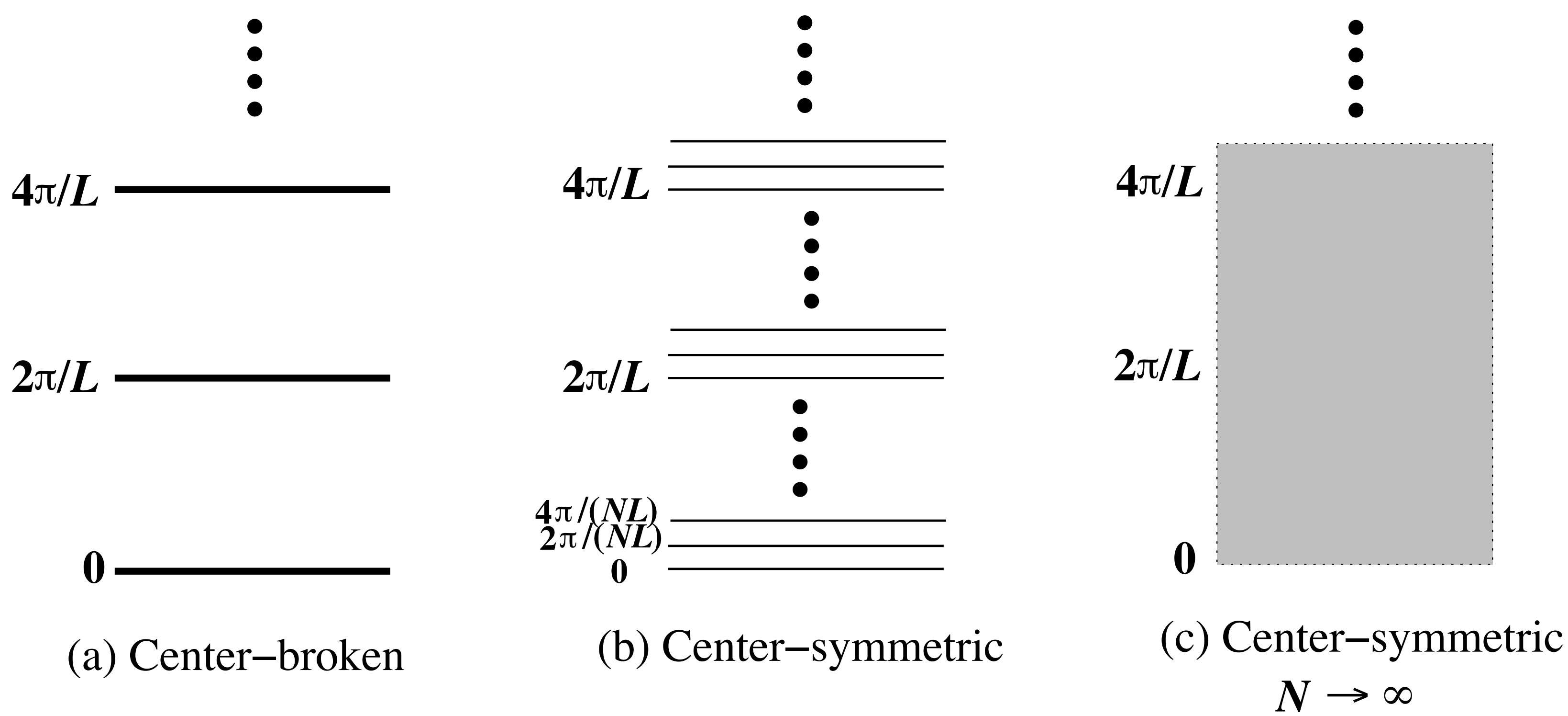
MÜ 07



thermal circle

Linde 80, magnetic scale

The dependence of perturbative spectrum to Wilson line holonomy



Long distance

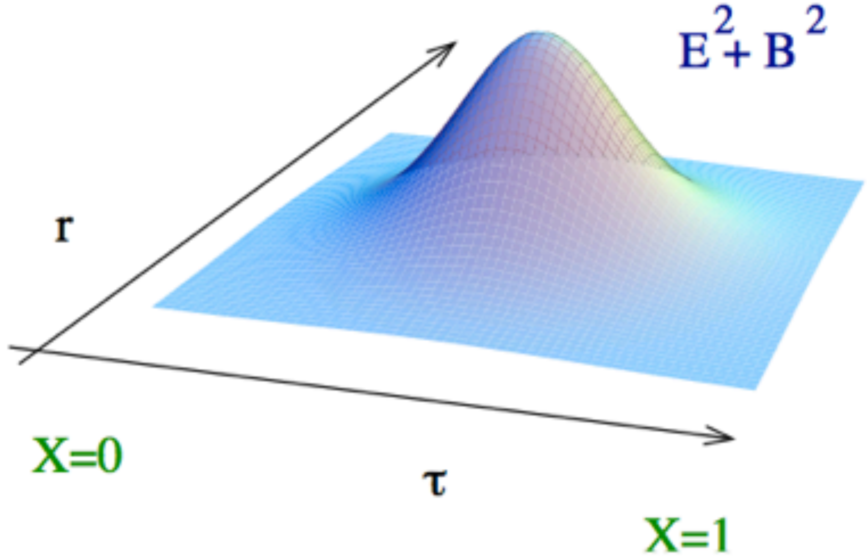
NP-Incalculable

NP-Calculable

NP-Incalculable (hope?)

Periodic instantons (calorons)

Instanton solution in R^4 can be extended to solution on $R^3 \times S^1$



$$Q_{top} = \pm 1$$

$$P_\infty = 1 \quad Q_M^\alpha = 0$$

$SU(2)$ solution has $1 + 3 + 1 + 3 = 8$ bosonic zero modes

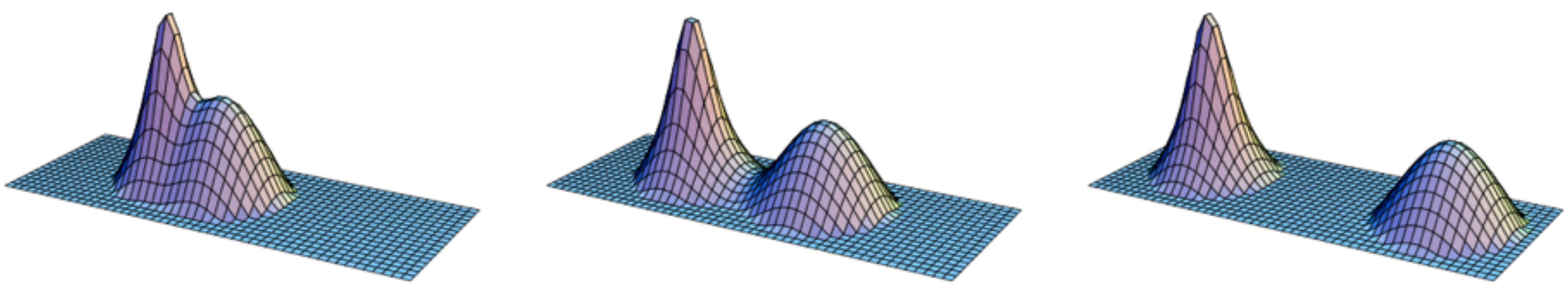
$$\int \frac{d\rho}{\rho^5} \int d^3x\, dx_4 \int dU\, e^{-2S_0} \qquad 2S_0 = \frac{8\pi^2}{g^2}$$

$4n_{adj}$ fermionic zero modes

$$\int d^2\zeta d^2\xi$$

Calorons at finite holonomy: monopole constituents

KvBLL (1998) construct calorons with non-trivial holonomy



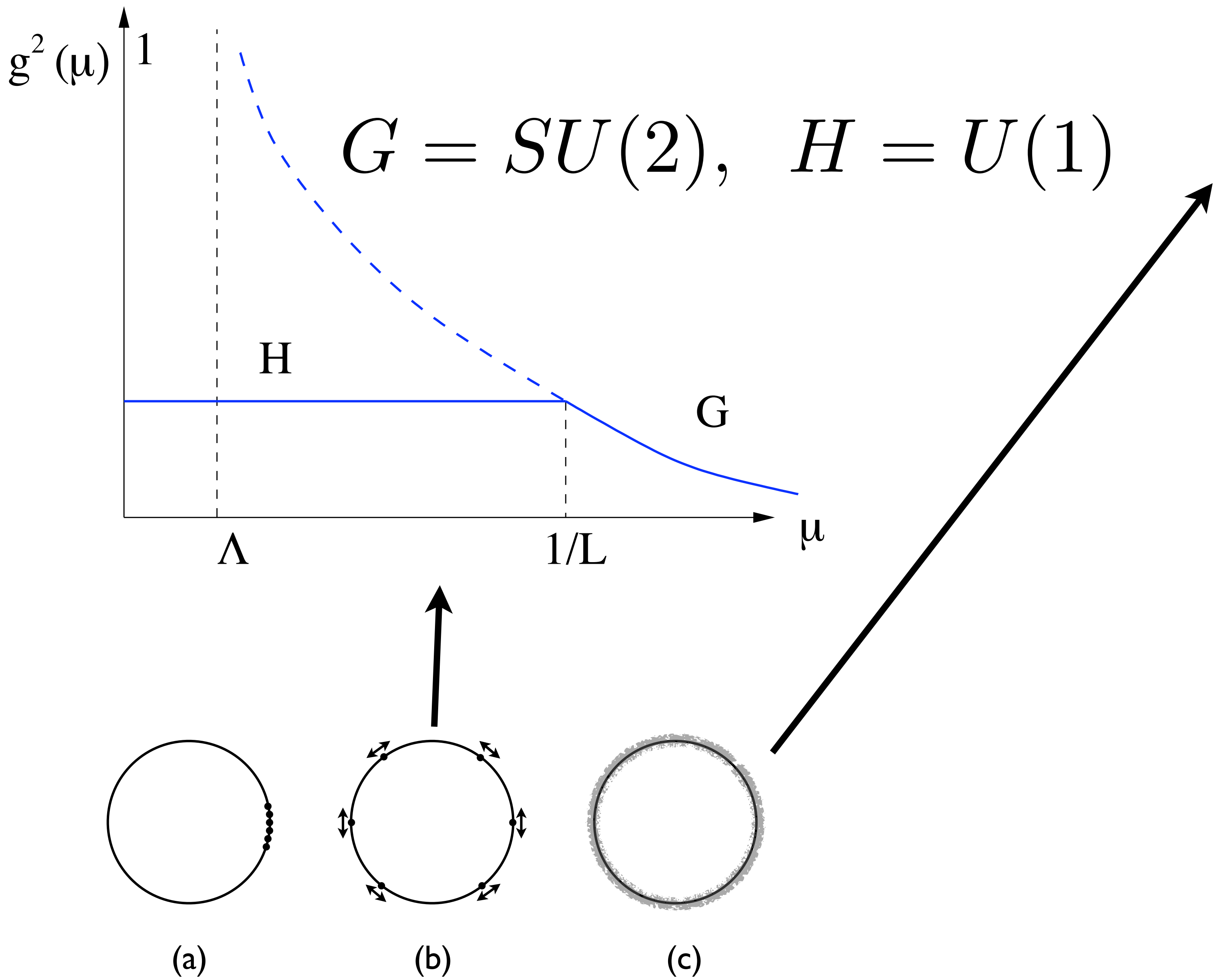
BPS and KK monopole constituents. Fractional topological charge, $1/2$ at center symmetric point.

$2 \times (3+1) = 8$ bosonic zero modes, 2×2 fermionic ZM.

$$\int d\phi_1 \int d^3x_1 \int d^2\zeta \, e^{-S_1} \int d\phi_2 \int d^3x_2 \int d^2\xi \, e^{-S_2}$$

Monopole-instantons: **van Baal, Kraan, (97/98), Lee-Lu (98) Lee-Yi (97).**
One of the most important realization in NP-QCD!

Trivializing monopole-instantons



The cleverness of van Baal et.al.:
To realize mon. instantons in the strong coupling regime with a weak coupling intuition!

With deformations and pbc, this is now simpler. At the time, quite non-trivial task.

We can now also understand what the role of these monopole-instantons etc. in the calculable regime.
This is the recent progress. (2007-......)

IR in perturbation theory is a free theory of "photons". Is this perturbative fixed point destabilized non-perturbatively?

Topological excitations in QCD(adj), SU(2), Nf=2

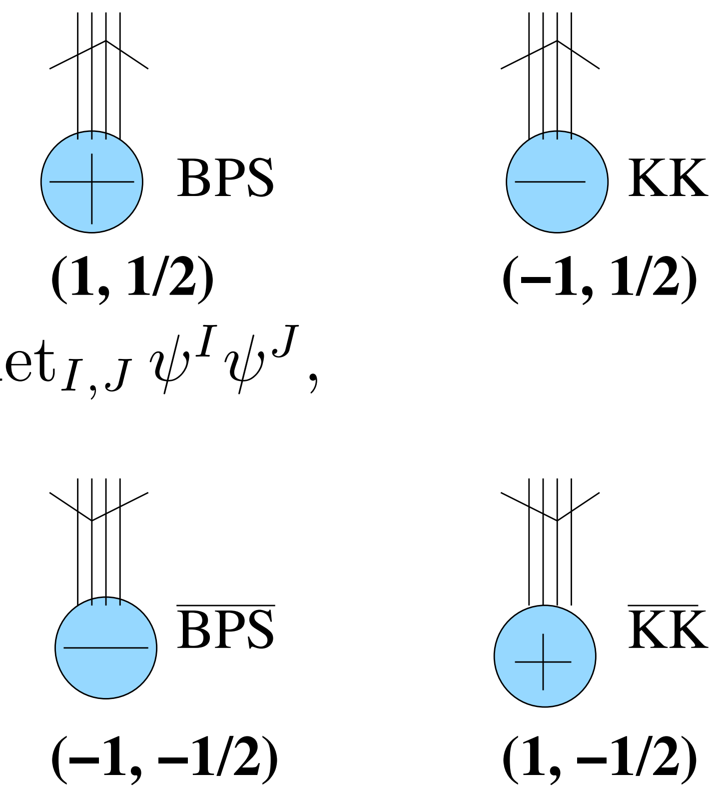
MÜ 2007

$$\left(\int_{S^2} F, \; \int_{R^3 \times S^1} F\tilde{F} \right)$$

index theorems
Callias 1978
E. Weinberg 1980
Nye-A.M.Singer, 2000
Poppitz, MU 2008
Atiyah-M.I.Singer 1975

Monopole-instantons

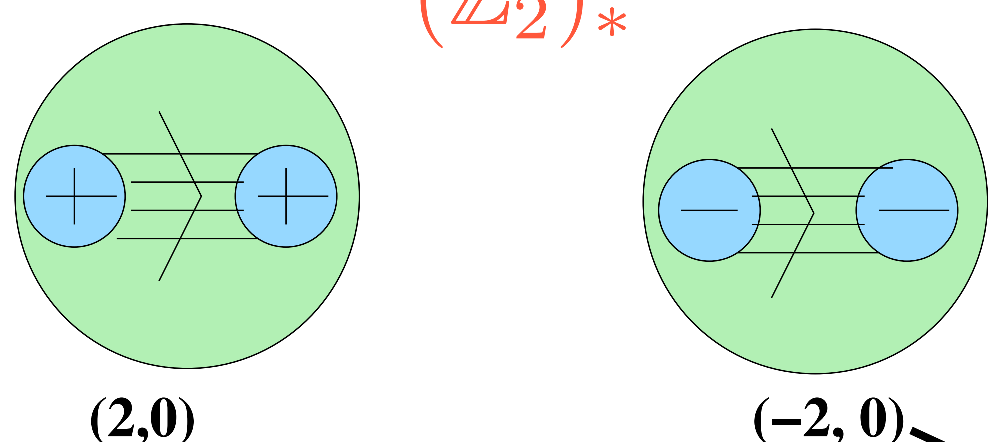
Magnetic Bions



$$e^{-S_0} e^{i\sigma} \det_{I,J} \psi^I \psi^J,$$

$$e^{-S_0} e^{i\sigma} \det_{I,J} \bar{\psi}^I \bar{\psi}^J$$

$(\mathbb{Z}_2)_*$



$$e^{-2S_0}\left(e^{2i\sigma} + e^{-2i\sigma}\right)$$

No net topological charge.

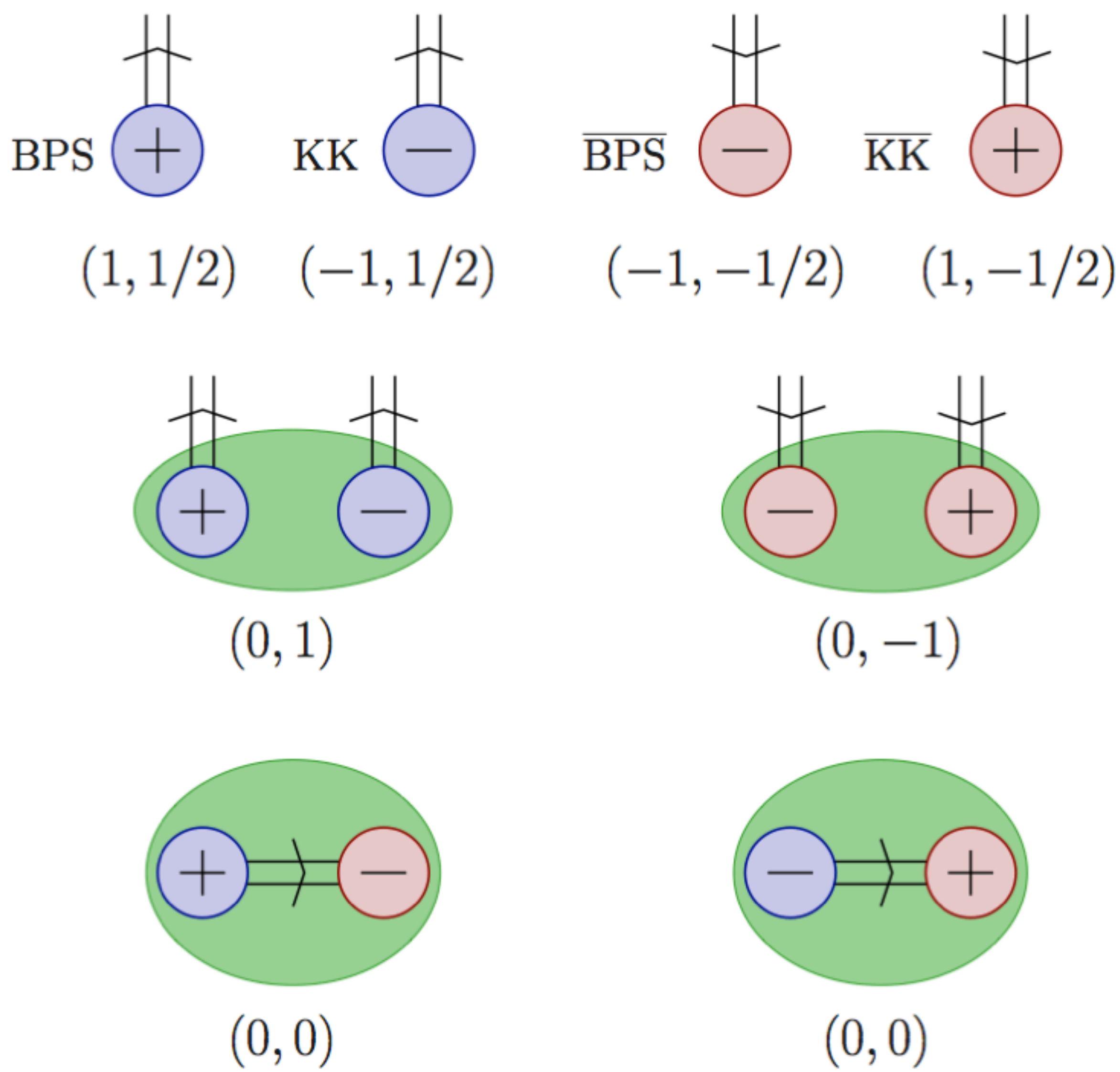
Discrete shift symmetry : $\quad \sigma \longrightarrow \sigma + \pi \qquad \psi^I \longrightarrow e^{i\frac{2\pi}{8}} \psi^I$

Crucial earlier work: van Baal, Kraan 97/98 and Lee, Lu, Yi, 97/98

Neutral bions: topologically and magnetically neutral!

Poppitz, Schaefer,MU, 2012

$$(Q_M, Q_{top}) = (\int_{S_2} B \cdot d\Sigma, \int_{R^3 \times S_1} F\tilde{F})$$



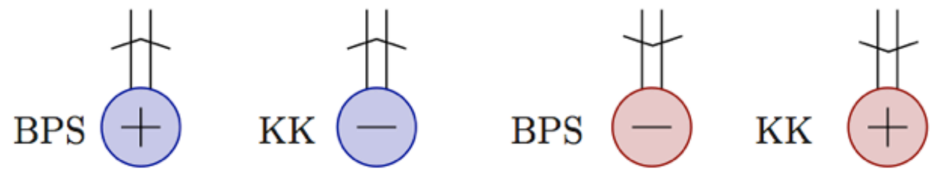
monopoles (monopole-instantons)

instantons

Neutral bions

Topological objects: Coupling to low energy fields

$$(Q_M, Q_{top}) = (\int_{S_2} B \cdot d\Sigma, \int_{R^3 \times S_1} F\tilde{F})$$



$e^{\mp b} e^{\pm i\sigma} (\lambda\lambda)$ $\quad$ $e^{\mp b} e^{\mp i\sigma} (\bar{\lambda}\bar{\lambda})$

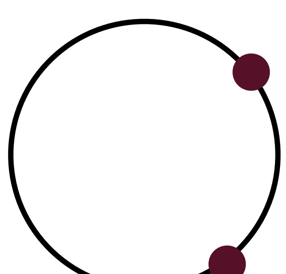
monopoles
mass gap for fermions

$(\lambda\lambda)(\lambda\lambda)$ $\quad$ $(\bar{\lambda}\bar{\lambda})(\bar{\lambda}\bar{\lambda})$

instantons
Anomaly

e^{-2b} $\quad$ e^{2b}

Neutral bions
Center stabilizing NP-potential!



$\Delta\theta$ $\qquad$ $b = \frac{4\pi}{g^2} \Delta\theta$

Effective potential for $m \neq 0$

Effective potential: molecules, monopoles, perturbation theory

Neutral and magnetic bion

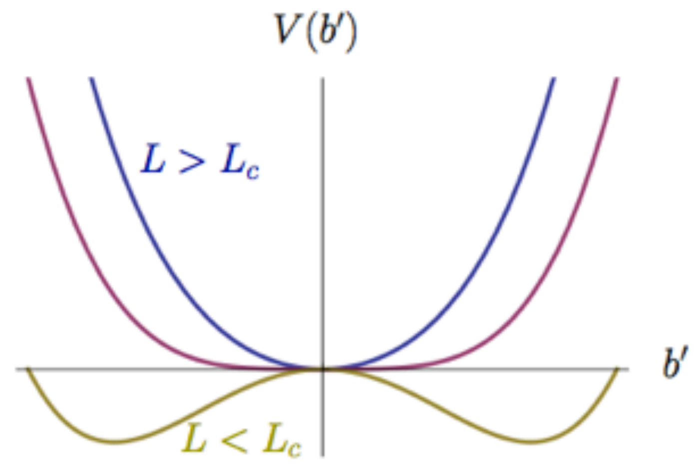
$$\tilde{V} = \cosh 2b' - \cos 2\sigma$$

Monopole-Instantons

$$+ \frac{\tilde{m}}{2\tilde{L}^2} \cos\sigma \left(\cosh b' - \frac{b' \sinh b'}{3 \log \tilde{L}^{-1}} \right)$$

$$- \frac{1}{1728} \left(\frac{\tilde{m}}{\tilde{L}^2} \right)^2 \frac{1}{\log^3 \tilde{L}^{-1}} (b')^2 \, .$$

One-loop P.T.



$\tilde{L} = L\Lambda$, $\tilde{m} = m/\Lambda$, $b' = \frac{4\pi}{g^2}(\Delta\theta - \pi)$

Critical S_1 size $\quad \tilde{L}_c^2 = \frac{\tilde{m}}{8}\left[1 + \mathcal{O}\left(\frac{1}{\log \tilde{L}}, \frac{\tilde{m}}{\tilde{L}^2}\right)\right]$,

Corresponds to $T_c = \sqrt{\frac{8}{\tilde{m}}} \Lambda_{QCD}$

Errors in the literature (for the sake of clarity, and with all due respect to the memory of D. Diakonov, I wish he was among us to discuss.)

1) Monopole-instatons lead to center stability in SYM.
Contrary, they do not even generate a potential for Wilson line. $\mathcal{M} \sim e^{-b+i\sigma}\lambda\lambda$

2) Calorons lead to center-stability. No, the important thing is the constituents of a caloron and their pairings. Caloron exist in the center-stable phase, but they do not even couple to Wilson line at leading order.

3) Monopole-instatons lead to center stability in YM: No reliable calculation can be done. Taking monopole-operators at face value, it leads to instability.

4) The use of the word "dyon" for monopole-instaton is a misnomer. Electric charge appears in Wilson line as

$$e^{iqA_4} \sim e^{iqb}.$$

The thing appearing in monopole operator is associated with breaking of dilatonic symmetry. (Jeff Harvey, 96) due to b-acquiring a vev, and it is a scalar dilatonic charge and not an electric charge.

NP ambiguity in semi-classical expansion:
Disaster or blessing in disguise?

Naive calculation of typical neutral defect amplitude, as you may guess as per QM example, multi-fold ambiguous!

As it stands, this is a **disaster!** Semi-classical expansion at higher order is void of meaning! In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. **The truth is far more subtler!**

NP-ambiguity in PT   Ambiguity in neutal-bions amplitude

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ii}]_\pm \,, \qquad (\text{up to } e^{-4S_0}) \qquad \mathrm{YM, CP(N-1)}$$

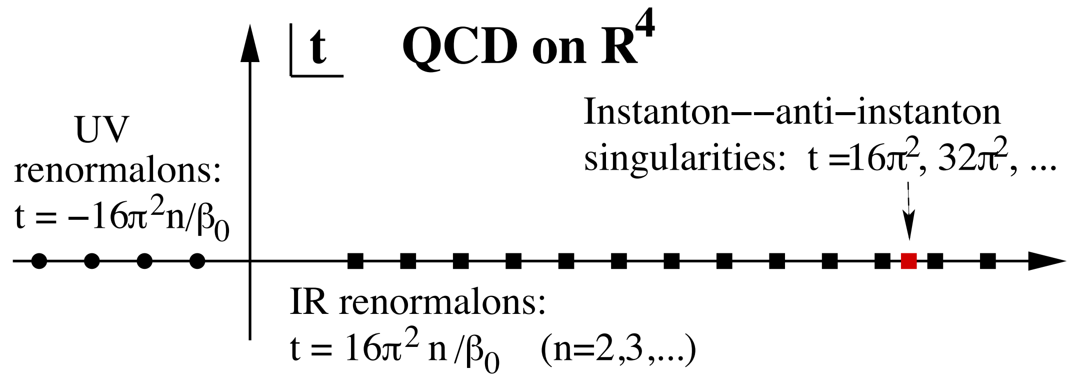
$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_\pm \,, \qquad (\text{up to } e^{-6S_0}) \qquad \mathrm{QCD(adj)}$$

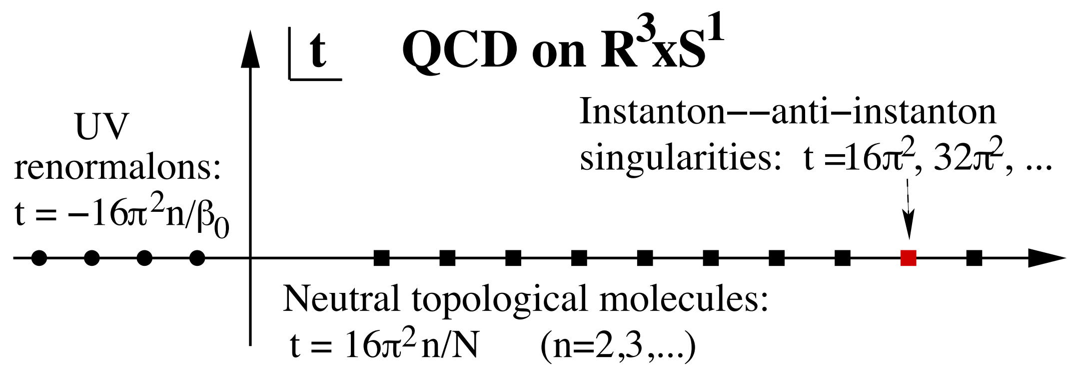
$$\mathrm{Im}[\mathcal{B}_{ii}]_\pm = \mathrm{Im}[\mathcal{M}_i\overline{\mathcal{M}}_i]_\pm \qquad \mathrm{Im}[\mathcal{B}_{ij}\overline{\mathcal{B}}_{ij}]_\pm = \mathrm{Im}[\mathcal{M}_i\overline{\mathcal{M}}_j\mathcal{M}_j\overline{\mathcal{M}}_i]_\pm$$

The ambiguities at order exp[-2S_I/N] cancel and
QFT is well-defined up to the ambiguities of order exp[-4S_I/N] !
Ambiguities in the IR-renormalon territory as per 't Hooft, David, Beneke,....

Semi-classical renormalons as neutral bions

Claim (with Argyres in 4d) and (with Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical realization of ‘t Hooft’s elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion. We showed this only at leading (but most important) order. Subleading orders underway.





More than three decades ago, ‘t Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that in fact, we can, at least, in the semi-classical regime of QFT.

Why is this happening?
Resurgence Theory and Transseries

Ecalle (1980s) formalized asymptotic expansion with exponentially small terms (called trans-series) & generalized Borel resummation for them by incorporating the Stokes phenomenon. Grand generalization of Borel-summability, a way to deal with non-Borel summable series. (the reason why 't Hooft moved away from this problem.)

Main result: Borel-Ecalle resummation of a transseries exists and is unique, if the Borel transforms of all perturbative series are all "endlessly continuable" =Set of all singularities on all Riemann sheets on Borel plane do not form any natural boundaries.
Such transseries are called "resurgent functions": Example of transseries:

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

Formal: perturbative + (non-perturbative) x (perturbative)

Resurgence theory in path integrals

Pham, Delabaere,....(1990s): Hamiltonian, some proofs. We wonder whether we can generalize this to path integrals of QM, because, path integral formulation generalize more easily to QFT.

Key step is in the analytic continuation of paths in field space (cf. Pham, and recent papers by Witten), to make sense of steepest descent and Stokes phenomenon in path integrals. (Lefschetz thimbles enter here first.)
(We actually use this implicitly, but need to make it more systematic.)
cf. a recent talk by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012.

Argyres, MU: QM in path integrals, similar to Kontsevich's ideas.

Graded Resurgence triangle

The structure of CP(N-1) and many QFTs is encoded into the following construct:

pert. theory around pert. vacuum

monopole-inst.
x (pert. fluctuations)

bions x (pert. fluctuations)

$$
\begin{array}{ccccccccc}
 & & & & f_{(0,0)} & & & & \\
 & & & e^{-\frac{A}{\lambda}+i\frac{\tilde{\Theta}_k}{N}} f_{(1,1)} & & e^{-\frac{A}{\lambda}-i\frac{\tilde{\Theta}_k}{N}} f_{(1,-1)} & & & \\
 & & e^{-\frac{2A}{\lambda}+2i\frac{\tilde{\Theta}_k}{N}} f_{(2,2)} & & e^{-\frac{2A}{\lambda}} f_{(2,0)} & & e^{-\frac{2A}{\lambda}-2i\frac{\tilde{\Theta}_k}{N}} f_{(2,-2)} & & \\
 & e^{-\frac{3A}{\lambda}+3i\frac{\tilde{\Theta}_k}{N}} f_{(3,3)} & & e^{-\frac{3A}{\lambda}+i\frac{\tilde{\Theta}_k}{N}} f_{(3,1)} & & e^{-\frac{3A}{\lambda}-i\frac{\tilde{\Theta}_k}{N}} f_{(3,-1)} & & e^{-\frac{3A}{\lambda}-3i\frac{\tilde{\Theta}_k}{N}} f_{(3,-3)} & \\
e^{-\frac{4A}{\lambda}+4i\frac{\tilde{\Theta}_k}{N}} f_{(4,4)} & & e^{-\frac{4A}{\lambda}+2i\frac{\tilde{\Theta}_k}{N}} f_{(4,2)} & & e^{-\frac{4A}{\lambda}} f_{(4,0)} & & e^{-\frac{4A}{\lambda}-2i\frac{\tilde{\Theta}_k}{N}} f_{(4,-2)} & & e^{-\frac{4A}{\lambda}-4i\frac{\tilde{\Theta}_k}{N}} f_{(4,-4)} \\
 & & \iddots & & \vdots & & \ddots & &
\end{array}
$$

No two column can mix with each other in the sense of cancellation of ambiguities.

Resurgence gives a far more refined classification of NP-saddles. Topology is blind in each column, resurgence can distinguish each row of a given column.

Topological classification is severely insufficient.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold. Examples are

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \ldots\Big)$$

Meaning, order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm \,, \qquad (\text{up to } e^{-4S_0})$$
$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$
$$0 = \ldots$$

Today on arxiv:

[5] arXiv:1308.1108 [pdf, other]
Resurgence theory, ghost-instantons, and analytic continuation of path integrals
Gokce Basar, Gerald V. Dunne, Mithat Unsal

[6] arXiv:1308.1115 [pdf, other]
Nonperturbative Ambiguities and the Reality of Resurgent Transseries
Inês Aniceto, Ricardo Schiappa

The latter gives an alternative derivation of our confluence equations by using resurgent analysis.

Decoding late terms in pert. theory.

This has a very deep implication in QFT

$$\mathrm{Disc}\,\mathbb{B}_{[0,0]} = -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \tag{1}$$

Using dispersion relation, we obtain

$$\begin{aligned} a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\ &= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \dots \right] \\ &+ O\left(\left(\frac{1}{4A}\right)^q\right) \end{aligned} \tag{2}$$

Late terms in
pert.exp. around
the pert. vac.

Early terms in pert.exp. around
neutral bion= 1/q corrections:

Neutral bion action

Exponentially suppressed
corrections: Bion-bion etc. terms.

Claim of resurgence theory: All orders perturbation theory knows the existence of all non-perturbative saddles, both those that the path integrals pass through and those that it does not!

Conclusions

Optimism.

Continuity and resurgence theory can be used in combination to provide a non-perturbative continuum definition of asymptotically free theories, and more general QFTs.

The construction may have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically.

I believe this puts the physical results about the quantum/thermal phase transitions I showed in the first three pages on a rigorous footing, at least in principle. This is most likely the reason why these results are almost verbatim with associated LGT result.