Unitary fermions: a review

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Outline

- Definitions and motivation
- Low temperature (ground state) properties
- Strategies for studying the system and summary of (some) results
- A few words on finite temperature and related systems
- Summary

Unitary fermions

A dilute mixture of spin 1/2 fermions at infinite two-particle scattering length

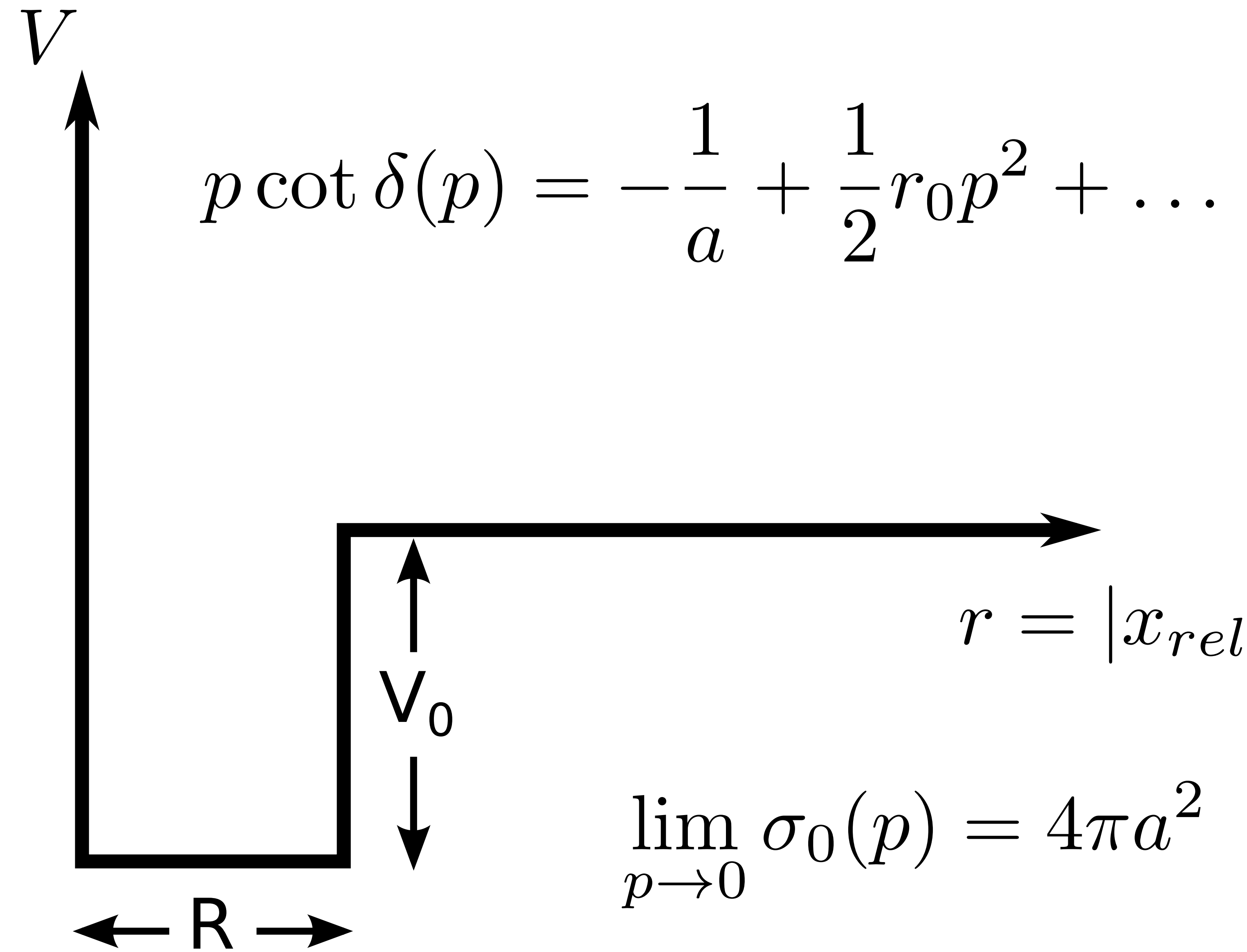
2-particle scattering from a short-range potential

$$\mathcal{A}(p) = \frac{4\pi}{M}\frac{1}{p\cot\delta(p) - ip}$$

$$p = \sqrt{EM}$$

scattering phase shift

$$p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}r_0 p^2 + \ldots$$



$$\lim_{p\to 0}\sigma_0(p) = 4\pi a^2$$

radial profile of spherically symmetric potential

2-particle scattering from a short-range potential

Unitary limit:

$$p \cot \delta = 0$$

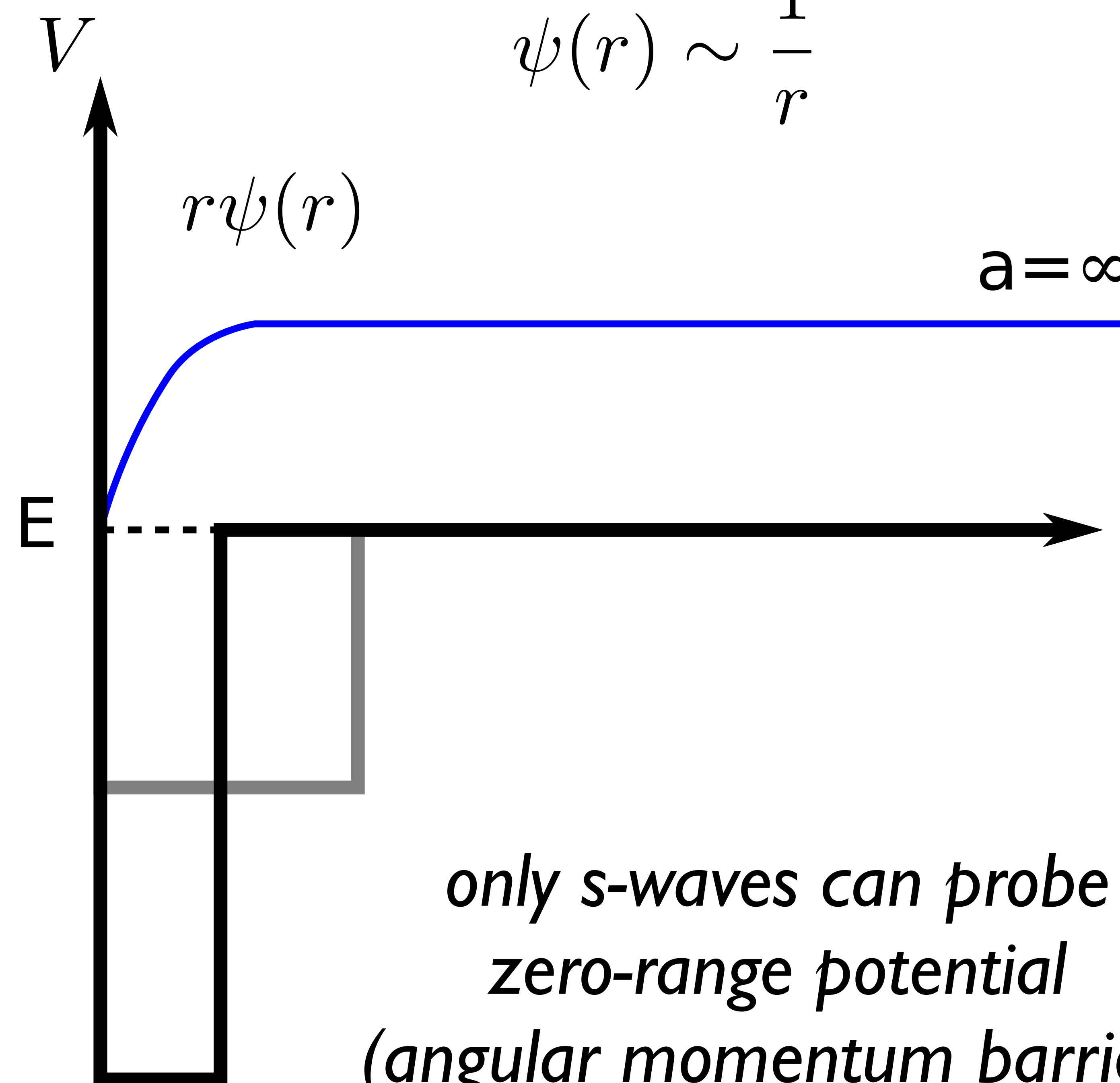
$$a \to \infty$$

$$R \to 0$$

$$\sigma_0 \leq \frac{4\pi}{p^2}$$

"unitarity bound"
is saturated!

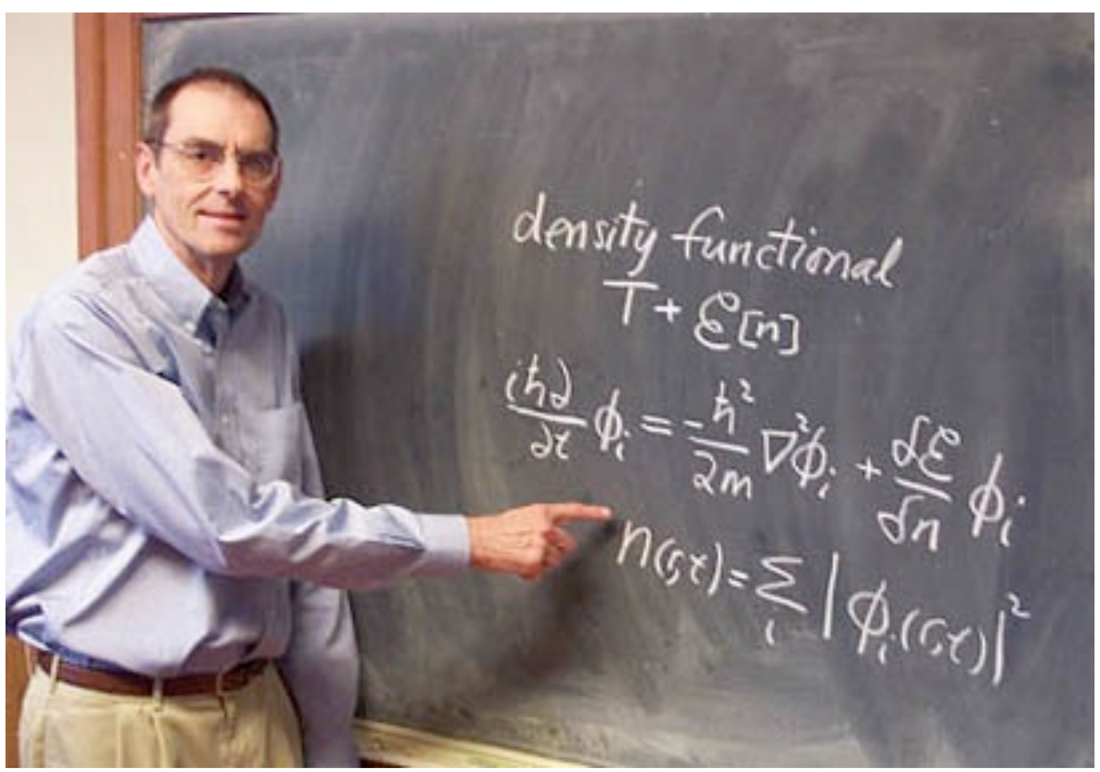
$$\psi(r) \sim \frac{1}{r}$$



only s-waves can probe zero-range potential (angular momentum barrier)

Many-body X challenge (circ. 1999)

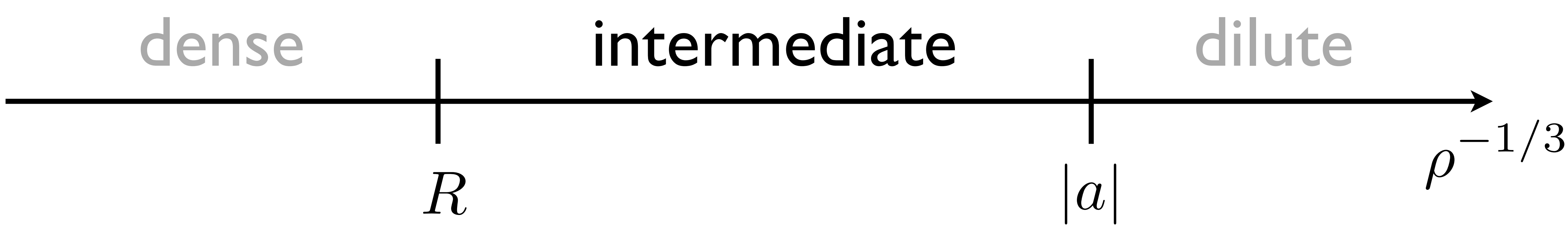
George Bertsch: What are the ground state properties of the many-body system composed of spin-1/2 fermions interacting via a zero-range, infinite scattering-length contact interaction?



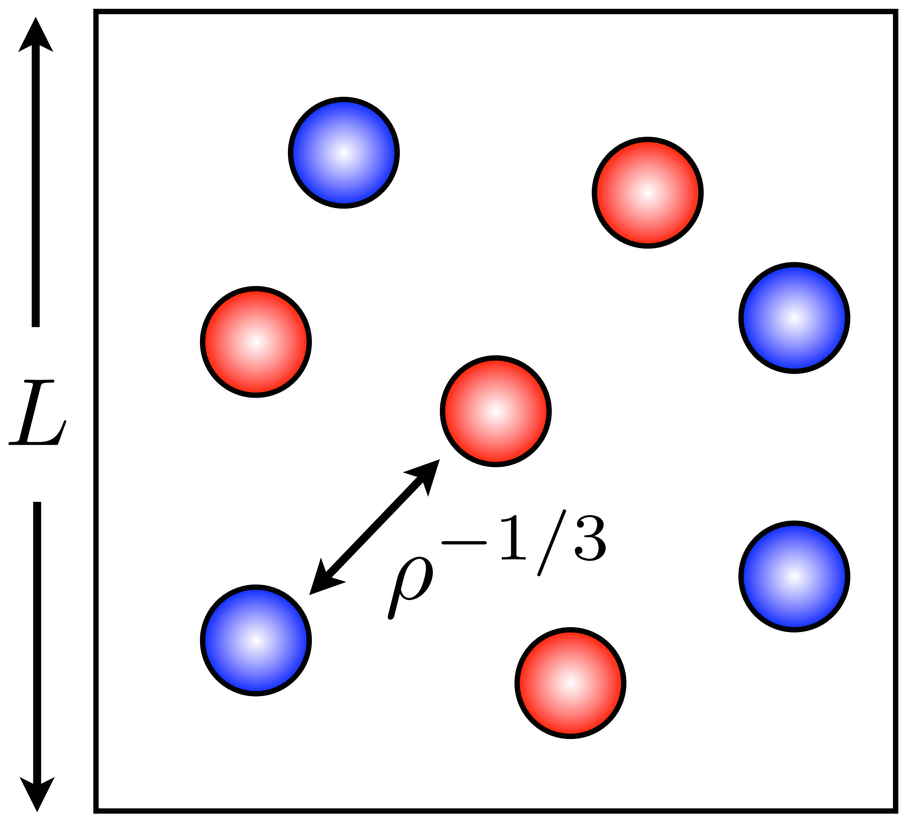
http://www.phys.washington.edu/users/bertsch/

- Original motivation: a model for dilute neutron matter
- Major advances on *all* fronts:
  - experiment
  - analytical calculation
  - numerical simulations

Interacting Fermi gas



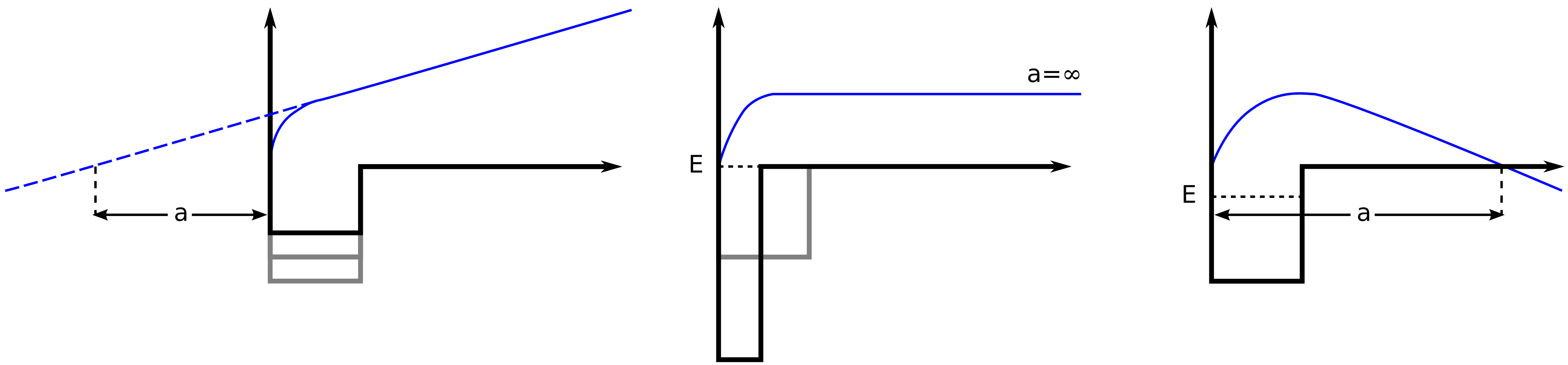
$$\rho = N/L^3$$



$$0 \leftarrow R \ll \rho^{-1/3} \ll a \to \infty$$

$$\psi \sim \frac{1}{r_{ij}} - \frac{1}{a}\,, \qquad r_{ij} = x_i^{\uparrow} - x_j^{\downarrow}$$

Interacting Fermi gas



BCS

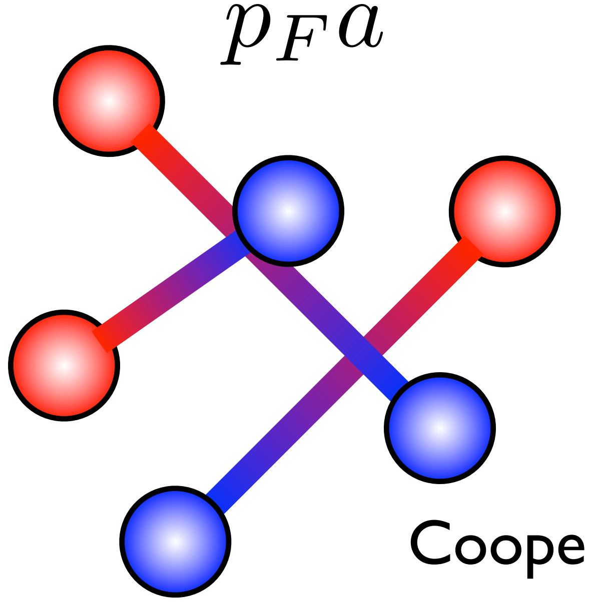
cross-over

BEC

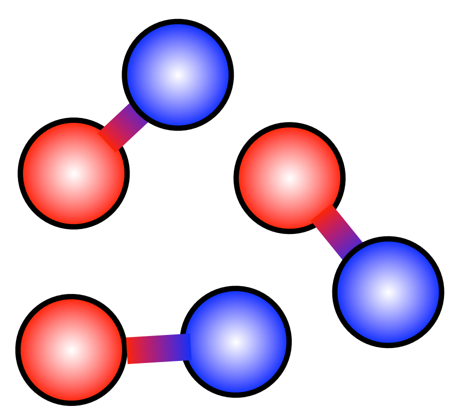
$$\frac{1}{p_F a} < 0$$

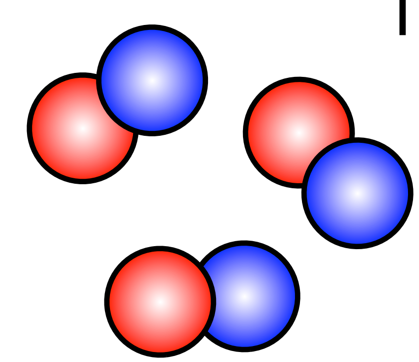
$$\frac{1}{p_F a} = 0$$

$$\frac{1}{p_F a} > 0$$



Cooper pairs





Tightly bound dimers

In the unitary limit, the properties of the system are universal in the sense that they are independent of the details of the interaction

Example 1: free energy (homogeneous)

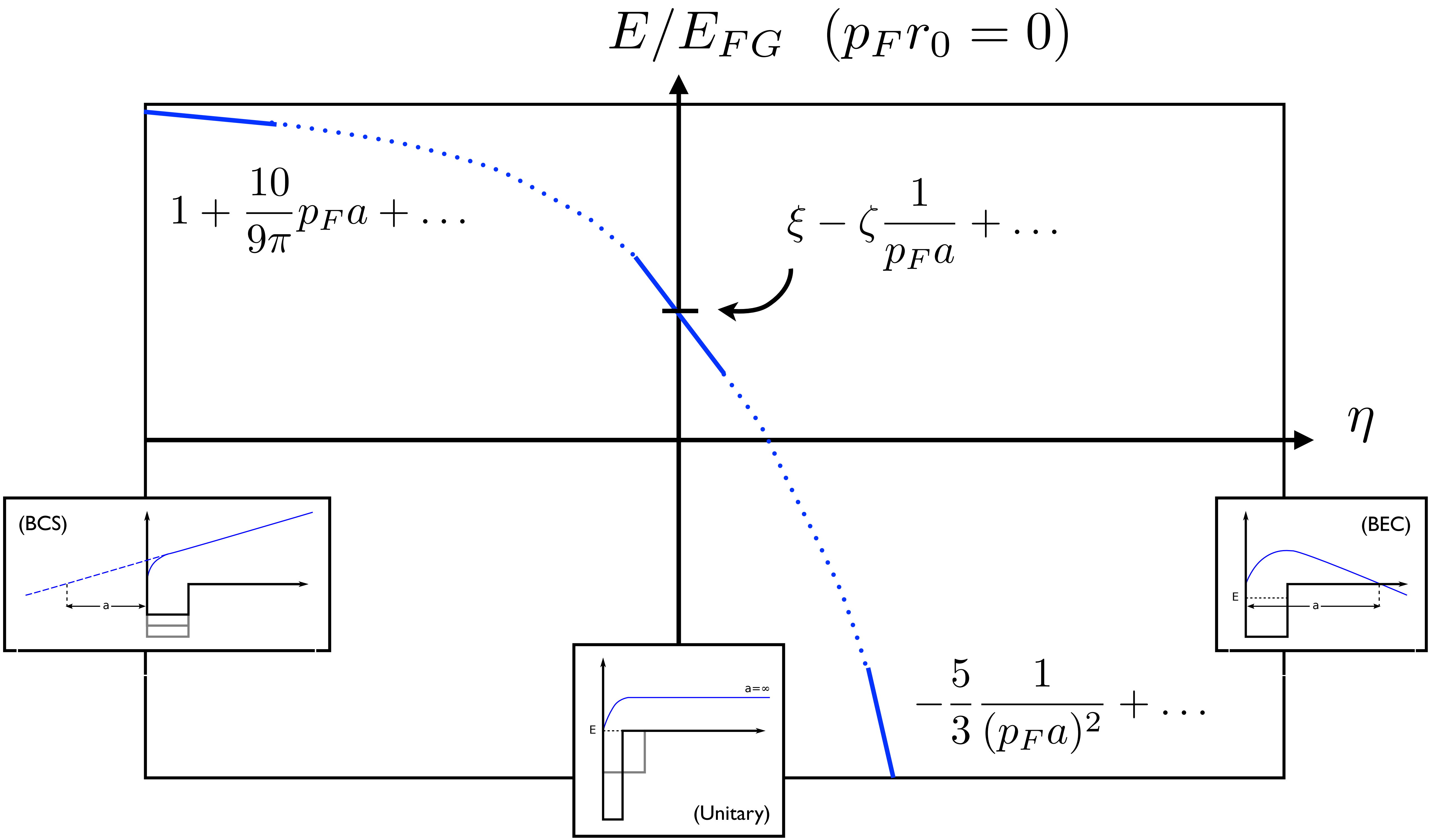
Observables are functions of two parameters:

$$F = E_{FG}\Xi(\eta, \theta)$$

$$E_{FG} = \frac{3}{5} N E_F \qquad \eta \equiv (p_F a)^{-1} \qquad \theta \equiv T/E_F$$

For this talk I'll take θ*=0*

Equation of state at zero temperature

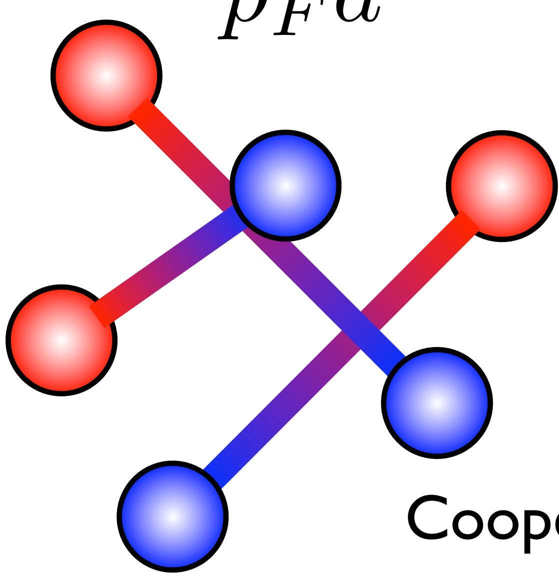


Example 2: pairing gap (homogeneous)

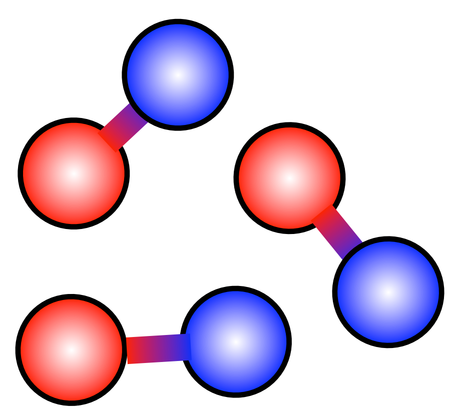
Δ = *half the energy required to split a fermion pair*

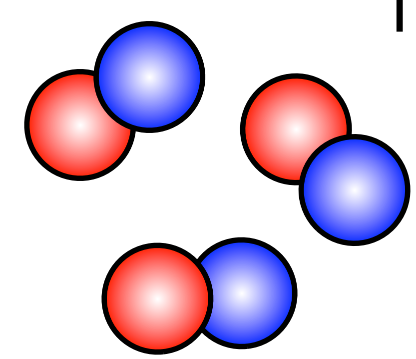
$$\frac{\Delta}{E_F} \sim e^{\frac{\pi}{2p_F a}} \qquad \frac{\Delta}{E_F} = \mathcal{O}(1) \qquad \frac{\Delta}{E_F} \sim \frac{1}{(p_F a)^2}$$

BCS	cross-over	BEC
$\frac{1}{p_F a} < 0$	$\frac{1}{p_F a} = 0$	$\frac{1}{p_F a} > 0$



Cooper pairs

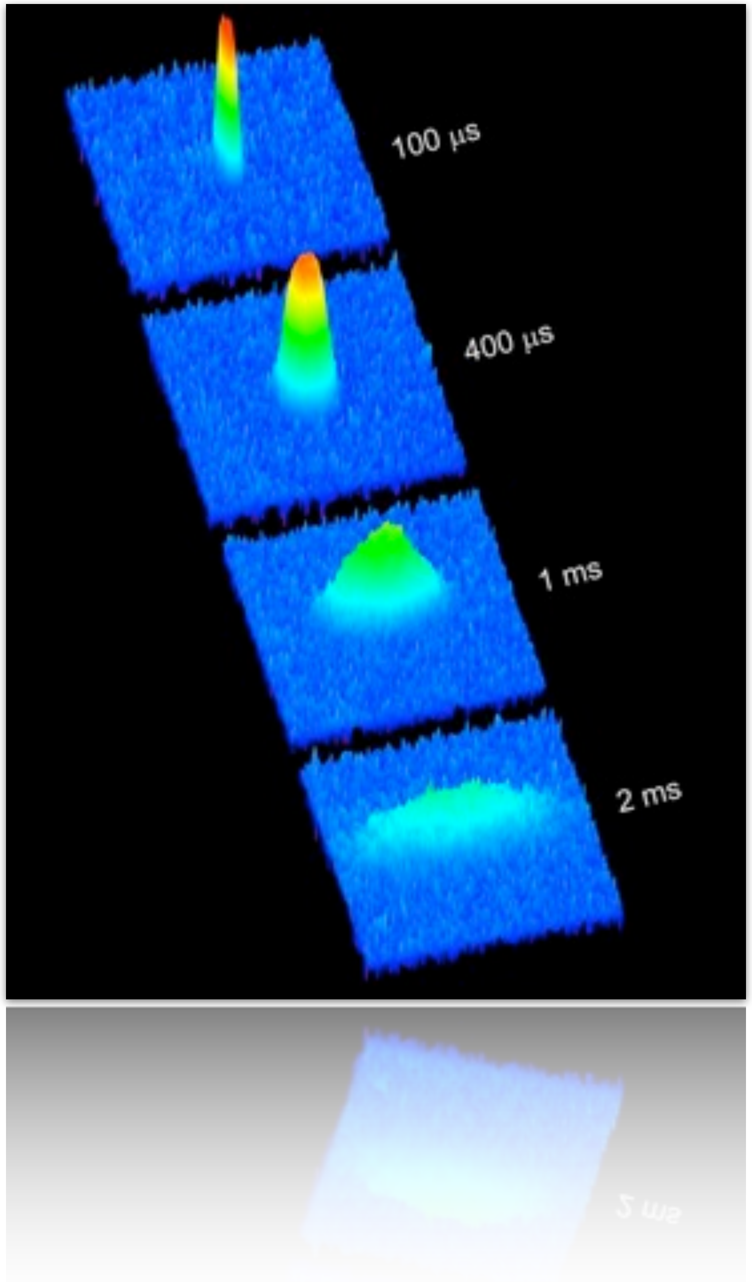




Tightly bound dimers

Physical realization: ultra-cold atoms

Science, **298**, pp. 2179-2182 (2002)



$$r_0({}^{40}K) \sim 60 a_0$$

(Bohr radius)

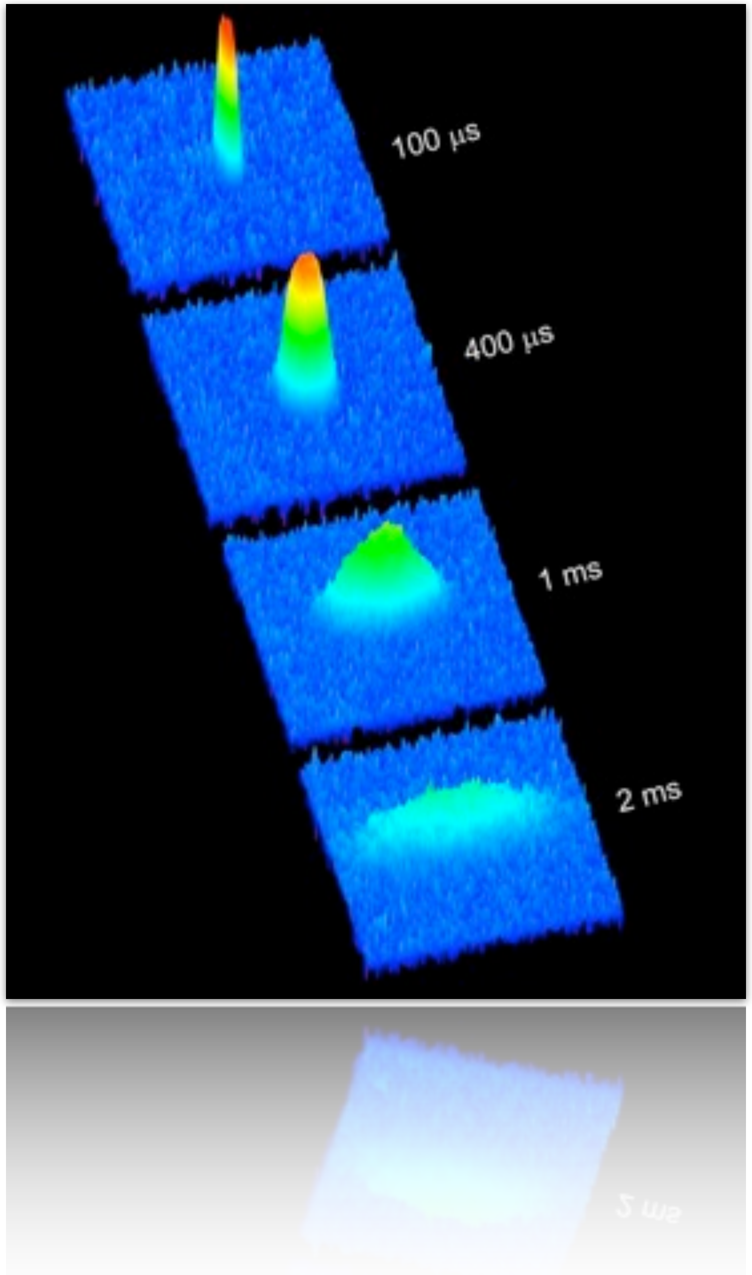
$$r_0({}^{6}Li) \sim 30 a_0$$

$$\rho^{-1/3} \sim 5000 - 10000 a_0$$

Interacting gas of fermions
(e.g., ^{6}Li or ^{40}K atoms)

Physical realization: ultra-cold atoms

***Science*, 298**, pp. 2179-2182 (2002)



$$r_0(^{40}K) \sim 60a_0$$

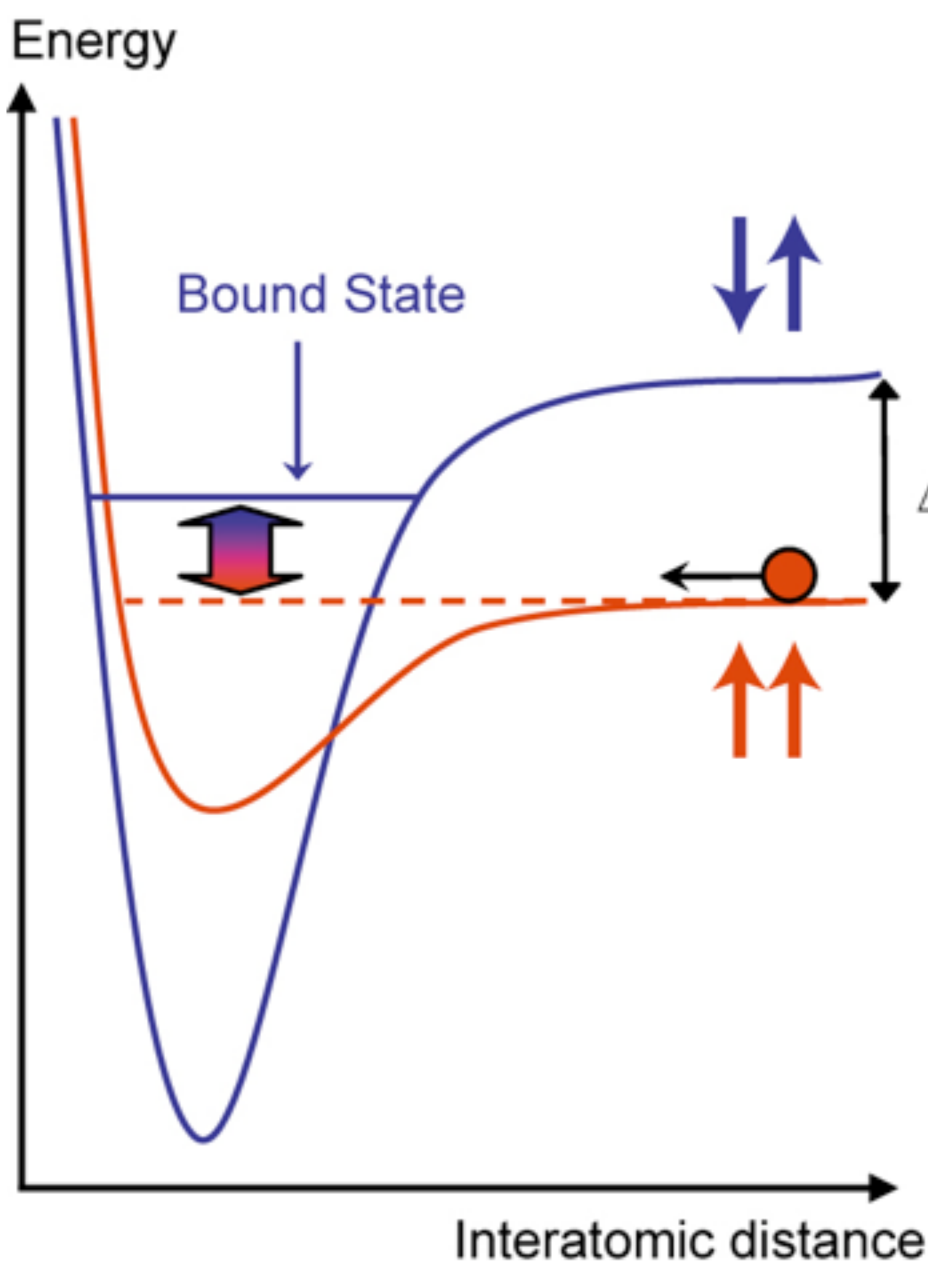
(Bohr radius)

$$r_0(^{6}Li) \sim 30a_0$$

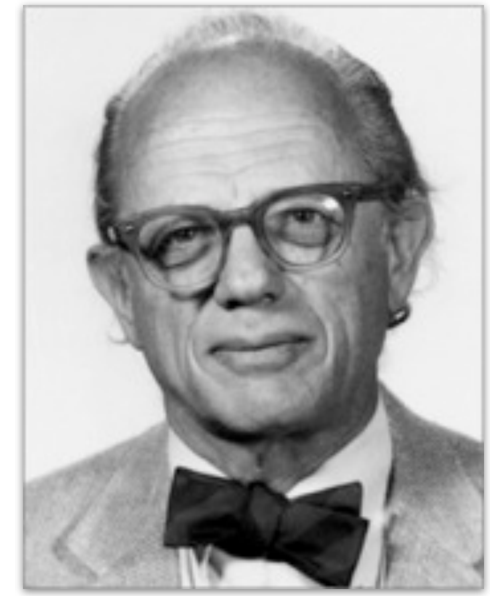
$$\rho^{-1/3} \sim 5000 - 10000a_0$$

Interaction strength tuned by exploiting properties of a Feshbach resonance

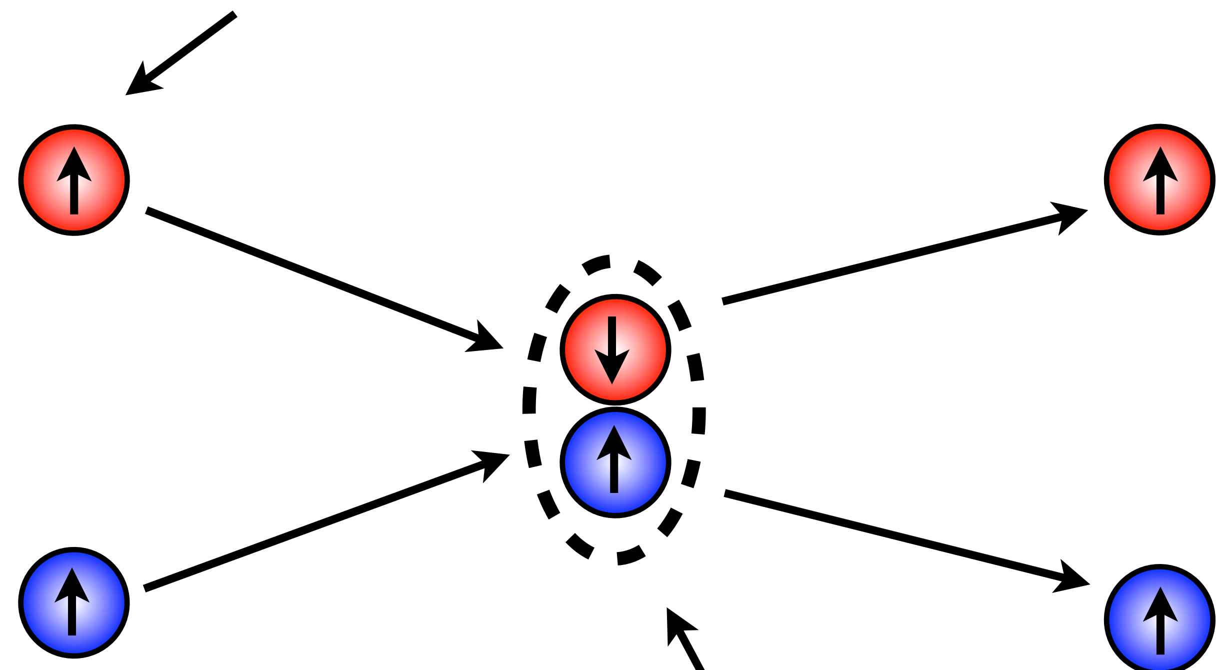
Feshbach resonances



http://cua.mit.edu/ketterle_group/

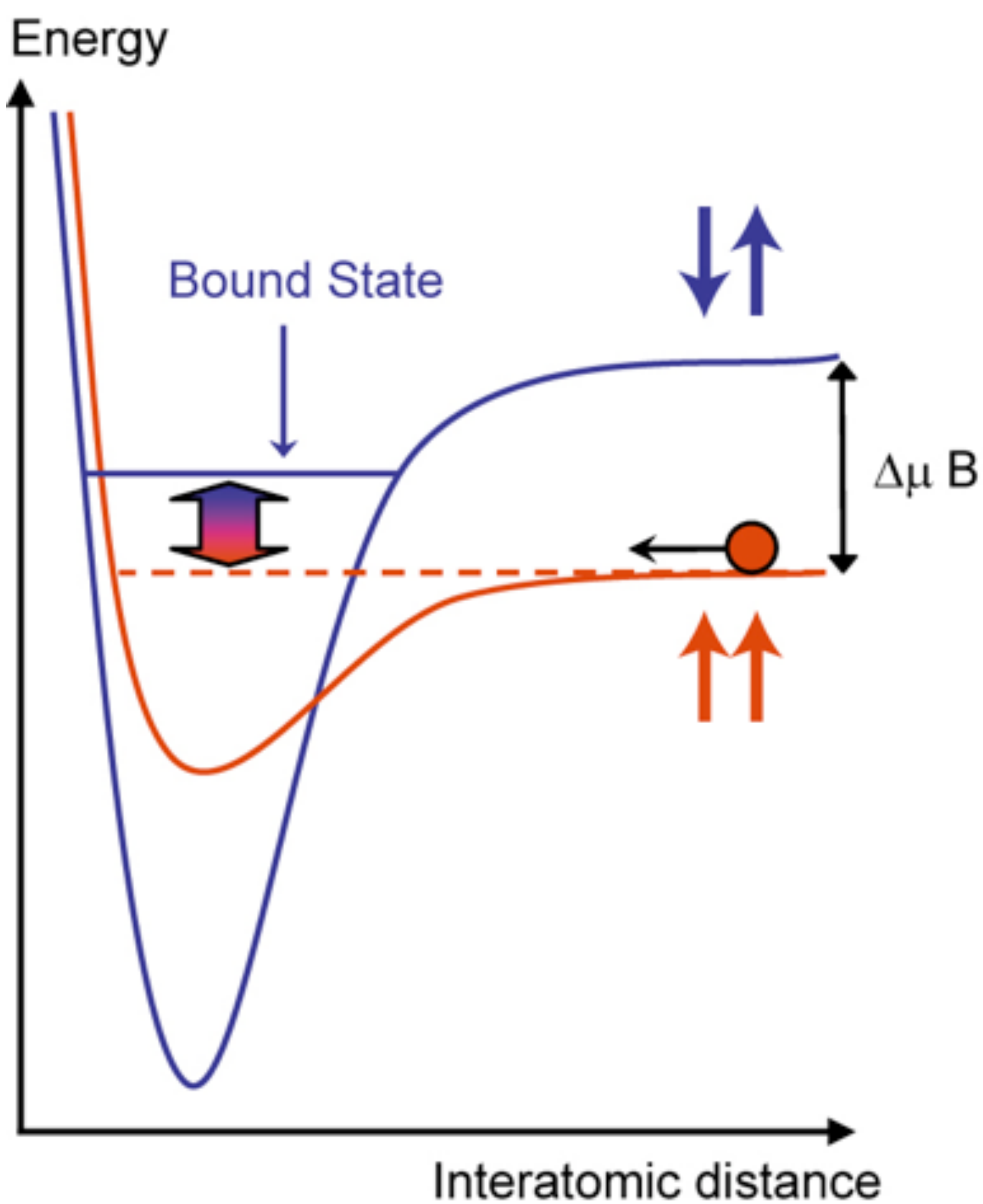


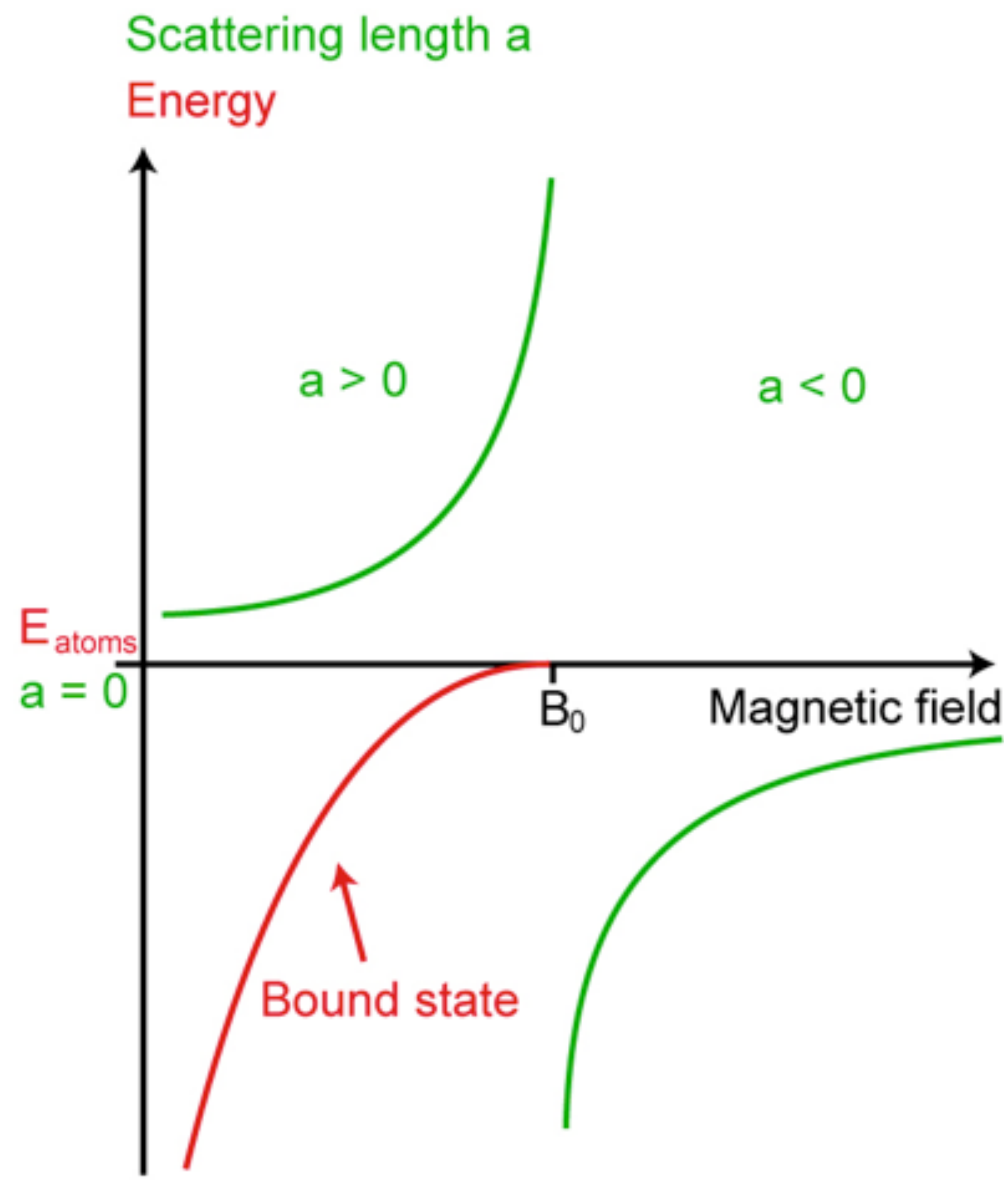
free unbound atoms



molecular bound state

Feshbach resonances





http://cua.mit.edu/ketterle_group/

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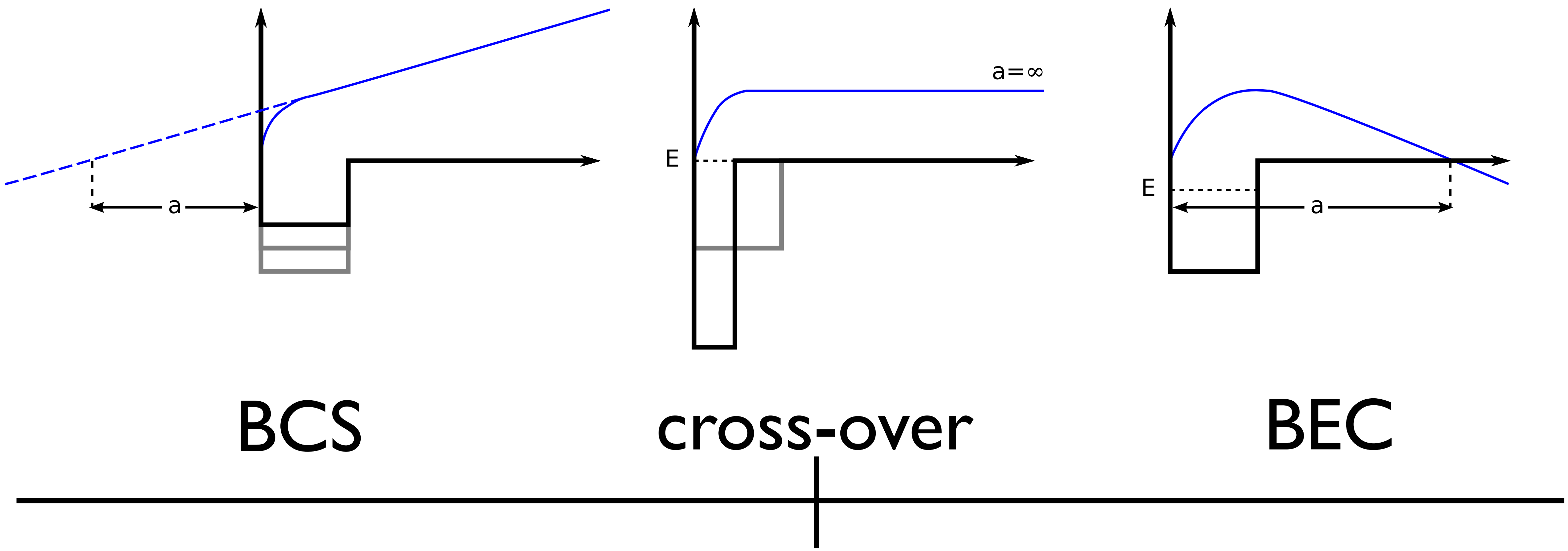
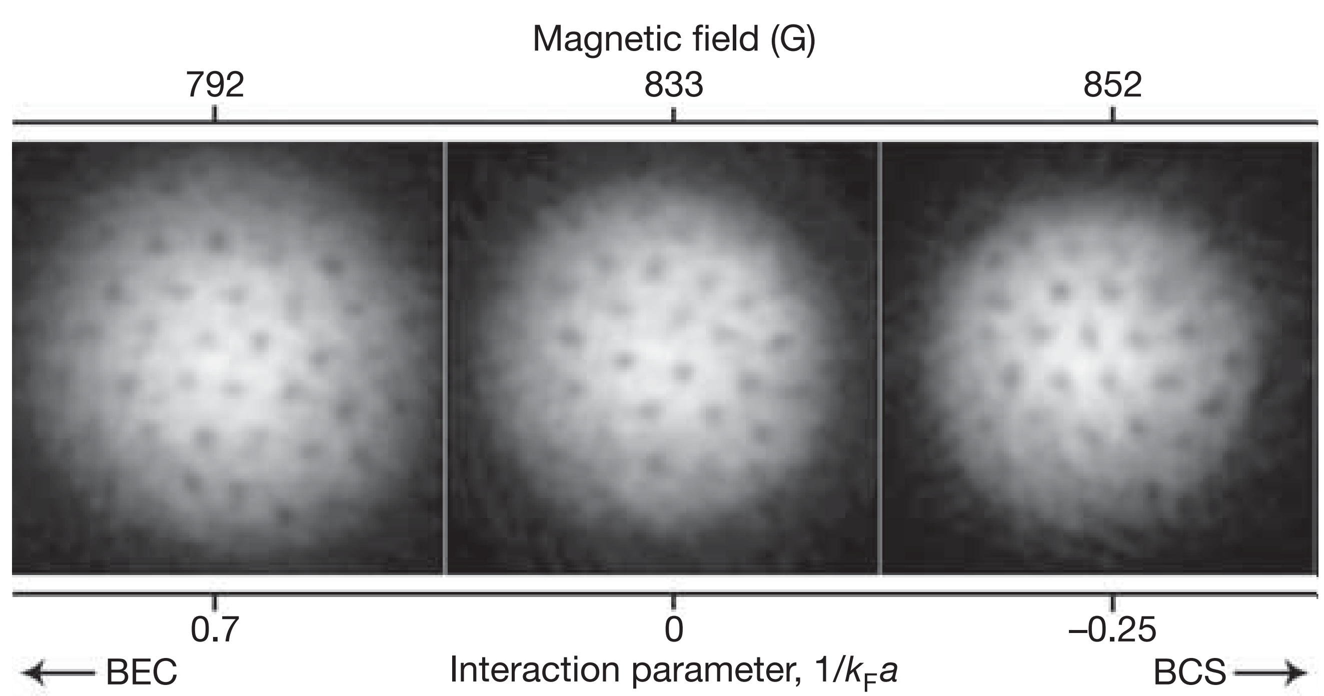
Example: direct observation of superfluidity

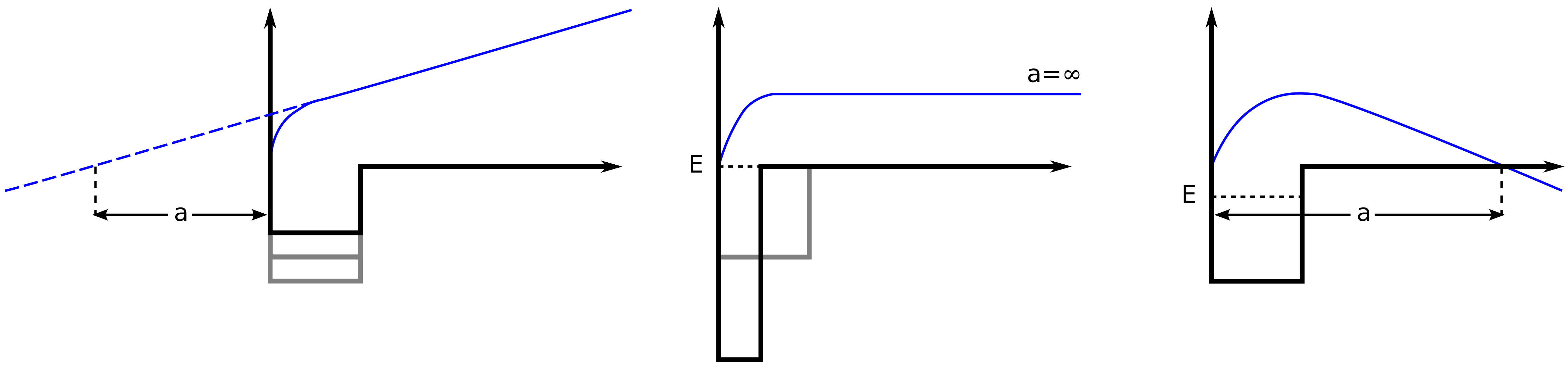


Vortex lattices observed in a strongly interacting, rotating Fermi gas of ^{6}Li

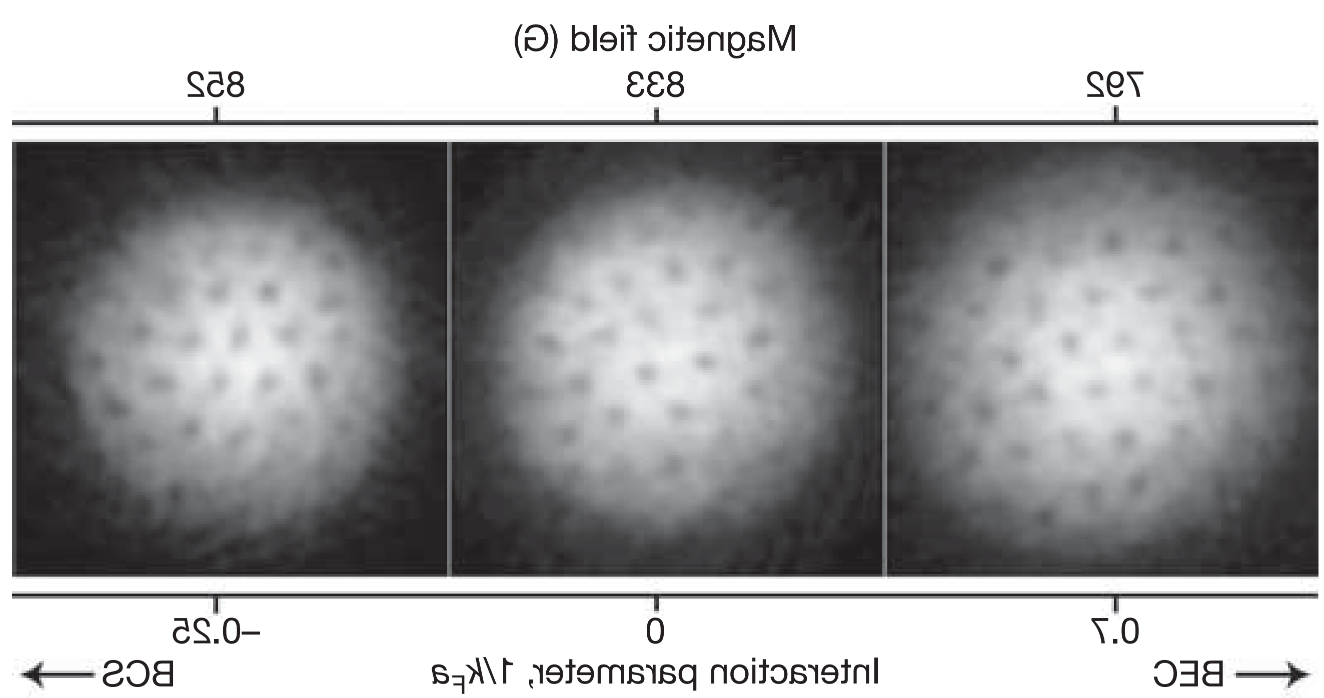
M.W. Zwierlein, *et al. (2005)*



Example: direct observation of superfluidity



BCS cross-over BEC

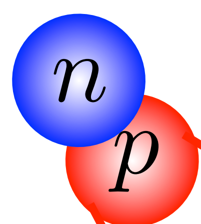


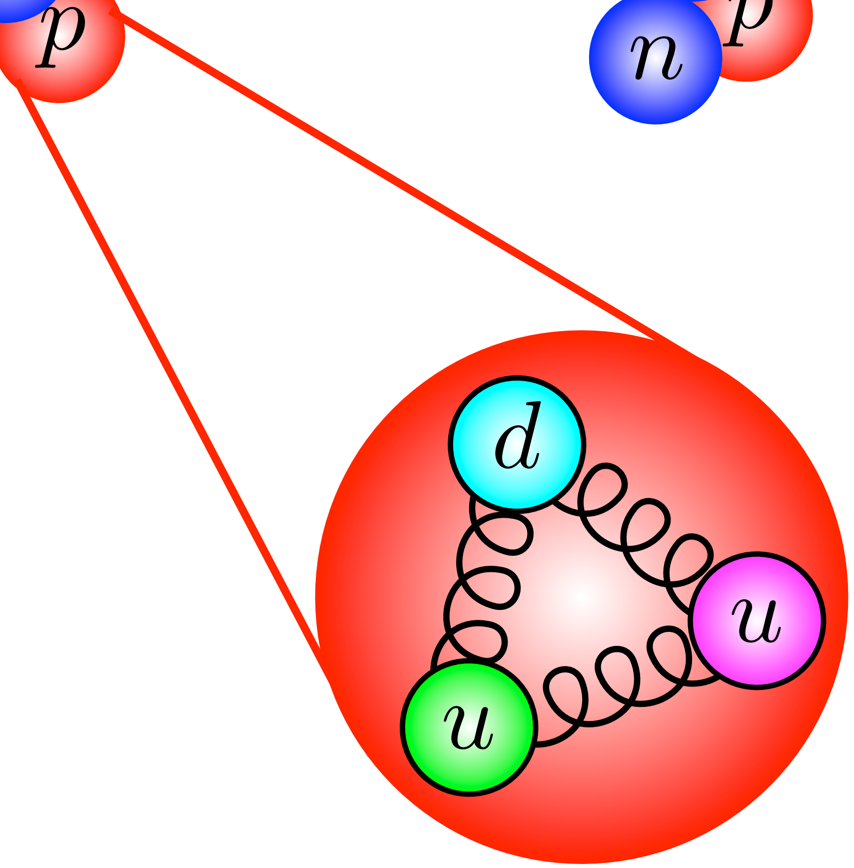
Vortex lattices observed in a strongly interacting, rotating Fermi gas of ^{6}Li

M.W. Zwierlein, *et al. (2005)*

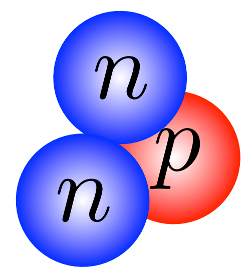
Nuclear physics: light nuclei

Deuteron

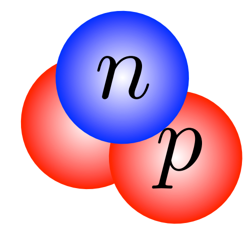




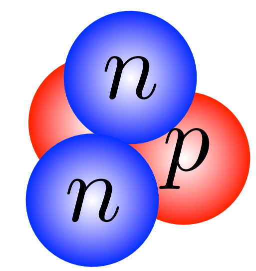
Triton



Helium-3



Alpha



$^1S_0(np)$ a ~ -23.7 fm

r_0 ~ 2.77 fm

$^3S_1(np)$ a ~ 5.4 fm

r_0 ~ 1.7 fm

(deuteron channel)

Unitary fermions can provide a starting point for an effective field theory description for nuclear physics

D. Lee (Aug. 6 @ 11am)

Effective field theory description

$$S = \int d^{D+1}x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu\right)\psi + C_0(\psi^\dagger\psi)^2\right]$$

$$\psi = (\psi_\uparrow, \psi_\downarrow)$$

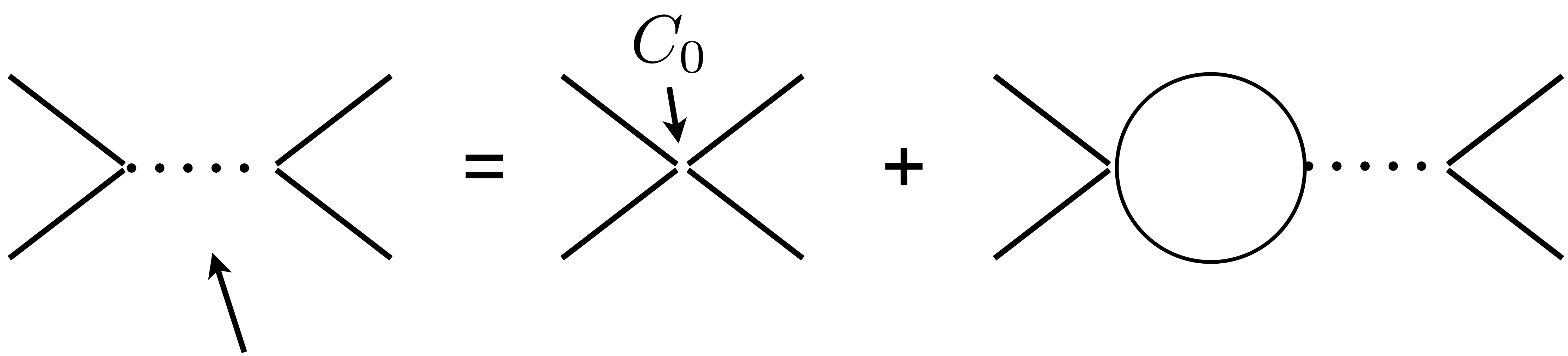
D=3 dimensions
(Euclidean space)

zero-range contact interaction

Effective field theory description

$$S = \int d^{D+1}x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu\right)\psi + C_0(\psi^\dagger\psi)^2\right]$$

Relationship between coupling (C_0) and scattering length:



two-particle
scattering amplitude

$$\mathcal{A}^{-1}(p)\big|_{p\to 0} = \frac{M}{4\pi a}$$

Effective field theory description

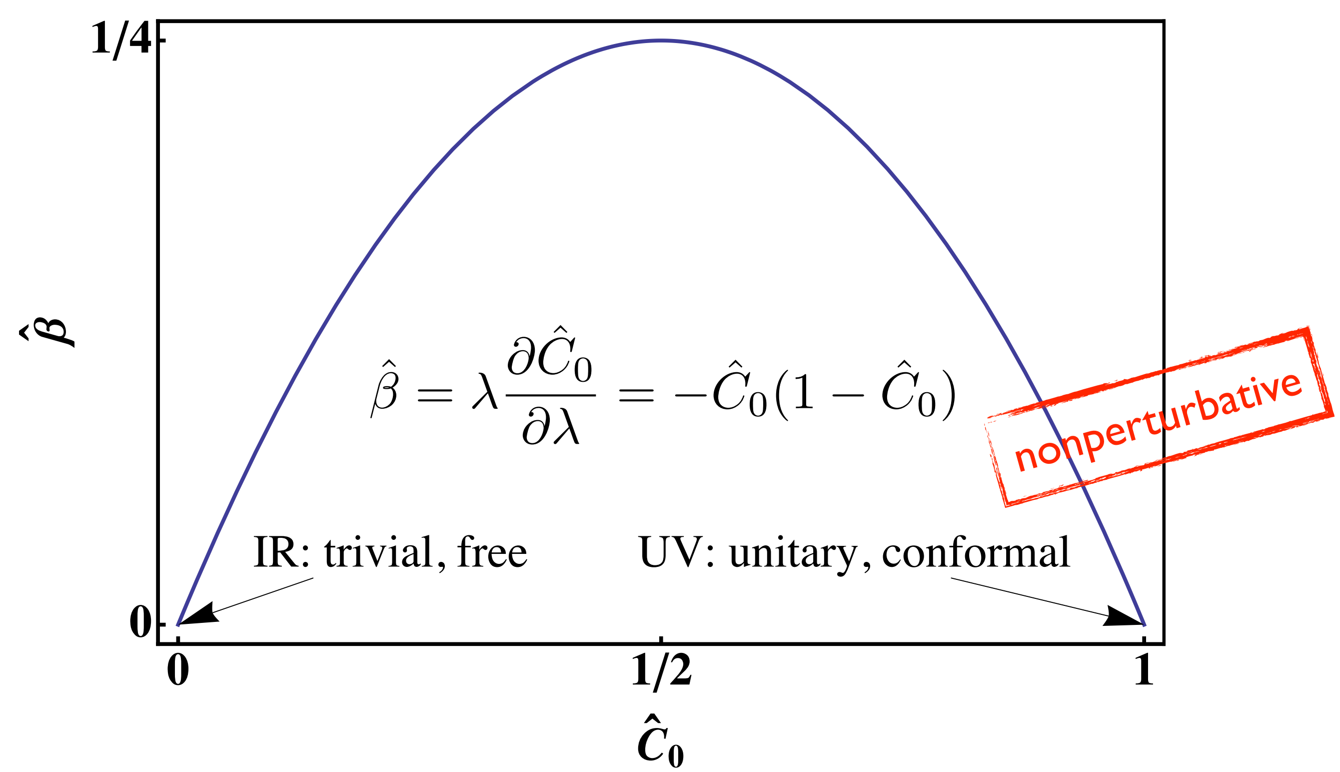
$$S = \int d^{D+1}x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu \right) \psi + C_0 (\psi^\dagger \psi)^2 \right]$$

Relationship between coupling (C_0) and scattering length:

$$\hat{C}_0(\lambda) = -\frac{M\lambda}{4\pi} C_0(\lambda)$$

Renormalization scale (λ)

D. B. Kaplan, *et al.* (1998)



Symmetries of free and unitary fermions

Conformal transformation:

$$x \to \frac{x}{1+\lambda t} \qquad t \to \frac{t}{1+\lambda t} \qquad \psi \to (1+\lambda t)^{3/2} e^{\frac{-iM\lambda x^2}{2(1+\lambda t)}} \psi$$

Scale transformation:

$$x \to \lambda x \qquad t \to \lambda^2 t \qquad \psi \to \lambda^{-3/2} \psi$$

(these, combined with rotations, translations, etc., are generated by the Schrödinger algebra)

T. Mehen, *et al.,* Phys. Lett. B **474** (2000) 145
Y. Nishida, *et al,* Phys. Rev. D **76** (2007) 086004

Virial theorems

$$H^{osc} = H + V$$

trapped Hamiltonian

untrapped Hamiltonian
(incl. interparticle potential)

harmonic potential

Conformal symmetry and *scale invariance* implies:

$$\langle H^n \rangle = \langle V^n \rangle$$ (for <u>any</u> trapped state, <u>any</u> n)

T. Mehen, Phys. Rev. A **78** (2008) 013614

Operator-state correspondence

(follows from conformal symmetry and scale-invariance)

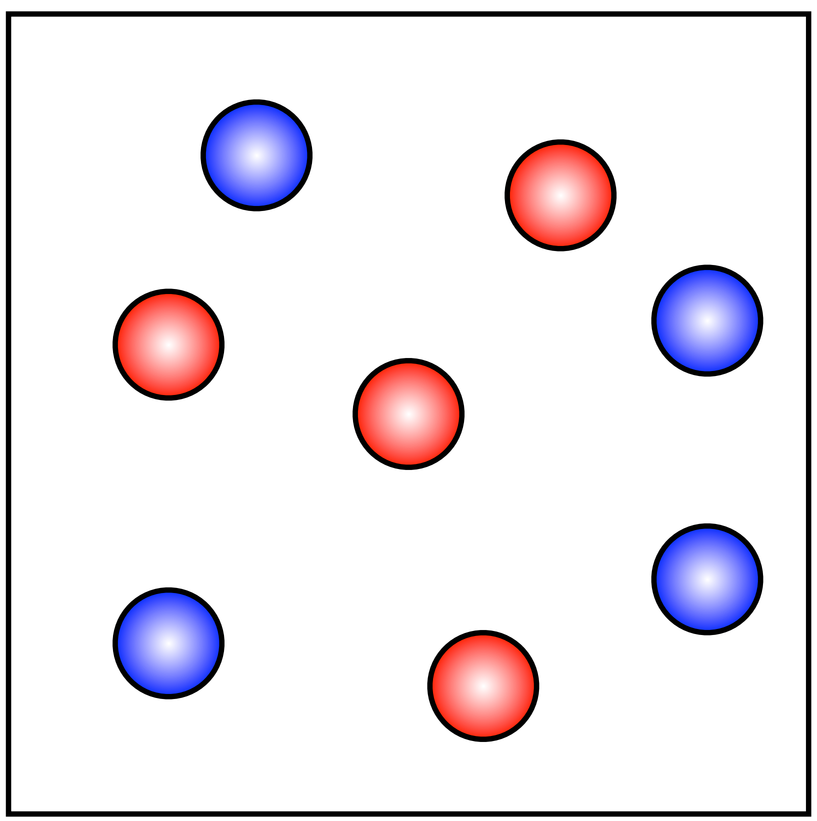
Y. Nishida and D.T. Son (2007)

"primary" operator constructed from N creation operators and having scaling dimension $\Delta_{\mathcal{O}}$

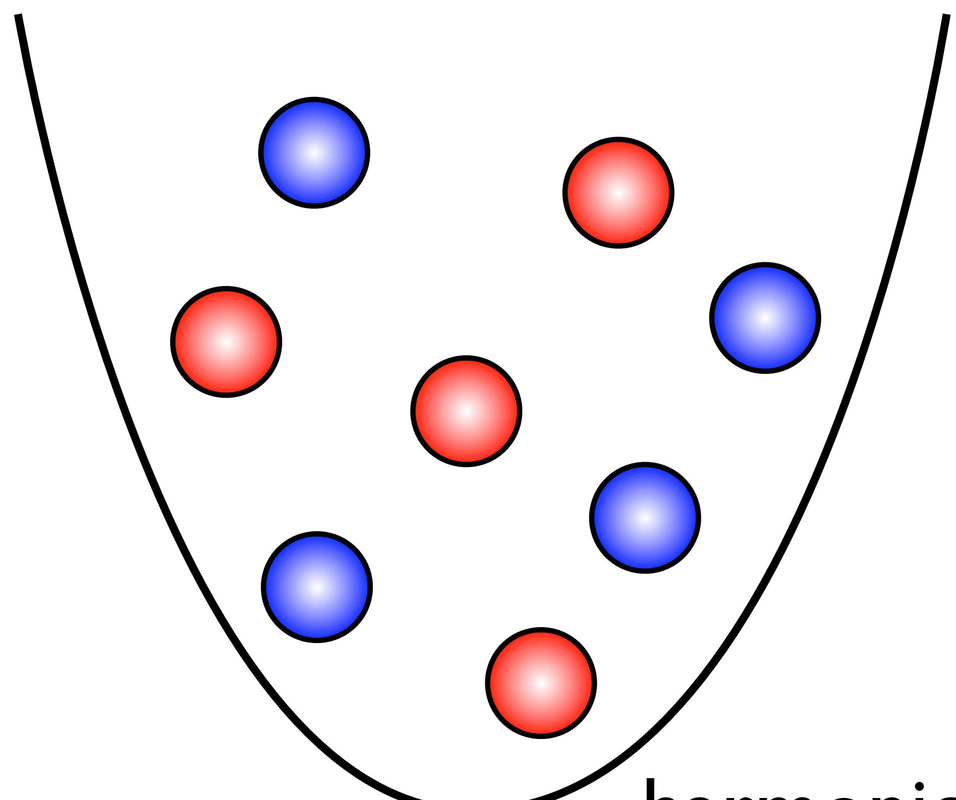
$$H^{osc} = H + V$$

$$|\Psi_{\mathcal{O}}\rangle = e^{-H/\omega}\mathcal{O}^{\dagger}|0\rangle$$

$$H^{osc}|\Psi_{\mathcal{O}}\rangle = \omega\Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$$



free space



harmonic trap

Operator-state correspondence

(follows from conformal symmetry and scale-invariance)

Y. Nishida and D.T. Son (2007)

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$$H^{osc} = H + V$$

$$|\Psi_{\mathcal{O}}\rangle = e^{-H/\omega}\mathcal{O}^\dagger|0\rangle$$

$$H^{osc}|\Psi_{\mathcal{O}}\rangle = \omega\Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$$

N	ℓ	$\mathcal{O}$	$\Delta_{\mathcal{O}}$	Reference
2	0	$\psi_\uparrow\psi_\downarrow$	2	S. Tan (2004)
3	0	$\psi_\uparrow\psi_\downarrow\partial_t\psi_\uparrow$	4.66622	S. Tan (2004)
3	1	$\psi_\uparrow\psi_\downarrow\nabla\psi_\uparrow$	4.27272	S. Tan (2004)
4	0	$\psi_\uparrow\psi_\downarrow\vec{\nabla}\psi_\uparrow\cdot\vec{\nabla}\psi_\downarrow$	5.1(1) 5.07(1)	S.Y. Chang, *et al.* (2007) J. von Stecher, *et al.* (2007)

Contact density

Provides a link between the <u>microscopic</u> properties of the system to the <u>macroscopic</u> properties of the system (examples on the next slides)

$$\mathcal{C}(x) = g^2 \langle \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\uparrow \psi_\downarrow(x) \rangle \qquad\qquad C = \int d^3x\, \mathcal{C}(x)$$

- Plays central role in universal (Tan) relations
  - away from unitarity
  - at finite temperature

Universal (Tan) relations

- First derivation (~2005) used generalized functions (distributions) to handle singularities in the zero-range limit:
  - S. Tan, Annals of Physics 323, 2952 (2008), cond-mat/0505200
  - S. Tan, Annals of Physics 323, 2971 (2008), cond-mat/0508320
  - S. Tan, Annals of Physics 323, 2987 (2008), arXiv:0803.0841
- Derivations based on quantum field theory methods (i.e., operator-product expansion):
  - E. Braaten and L. Platter, Phys. Rev. Lett. 100, 205301 (2008)
  - E. Braaten, D. Kang, and L. Platter, Phys. Rev. A 78, 053606 (2008), arXiv: 0806.2277

Universal (Tan) relations: microscopic

Fall-off of momentum distribution: $\boxed{C} = \lim_{k \to \infty} k^4 \rho_\sigma(k)$

Local pair density: $\boxed{\mathcal{C}} = \lim_{s \to 0} \frac{4}{s^4} N_{pair}(s)$

Number of fermion pairs in a small sphere of radius s
(computable from density-density correlator)

Universal (Tan) relations: macroscopic

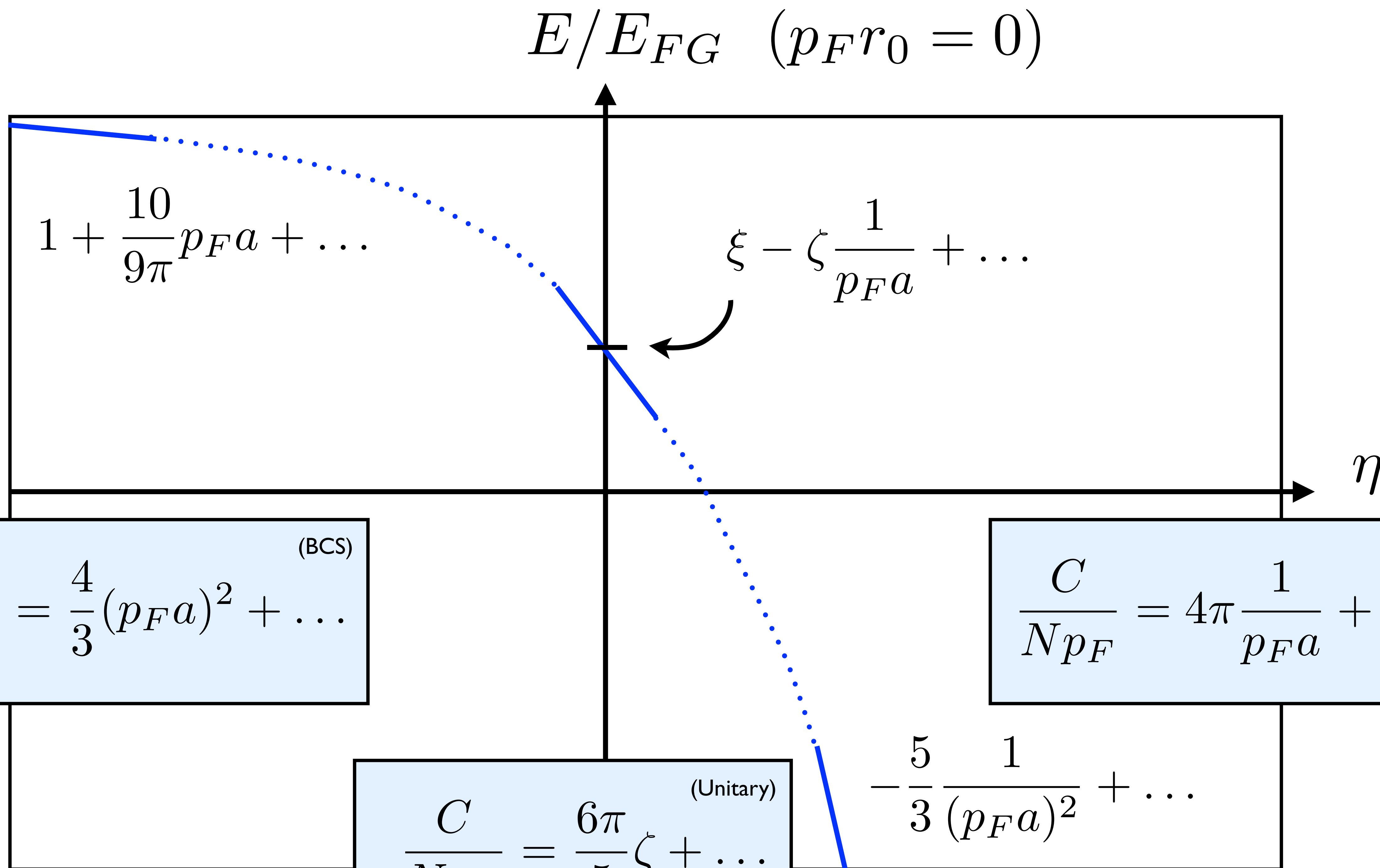
Generalized virial theorem: $$\langle H \rangle = \langle V \rangle - \frac{\boxed{C}}{8\pi M a}$$

Pressure relation: $$\mathcal{P} = \frac{2}{3}\mathcal{E} + \frac{\boxed{\mathcal{C}}}{12\pi M a}$$

Adiabatic sweep theorem: $$\boxed{C} = (-4\pi M)\frac{\partial E}{\partial a^{-1}}$$

$\vdots$ (and several more)

Dimensionless contact



(BCS)

$$\frac{C}{N p_F} = \frac{4}{3} (p_F a)^2 + \ldots$$

(BEC)

$$\frac{C}{N p_F} = 4\pi \frac{1}{p_F a} + \ldots$$

(Unitary)

$$\frac{C}{N p_F} = \frac{6\pi}{5} \zeta + \ldots$$

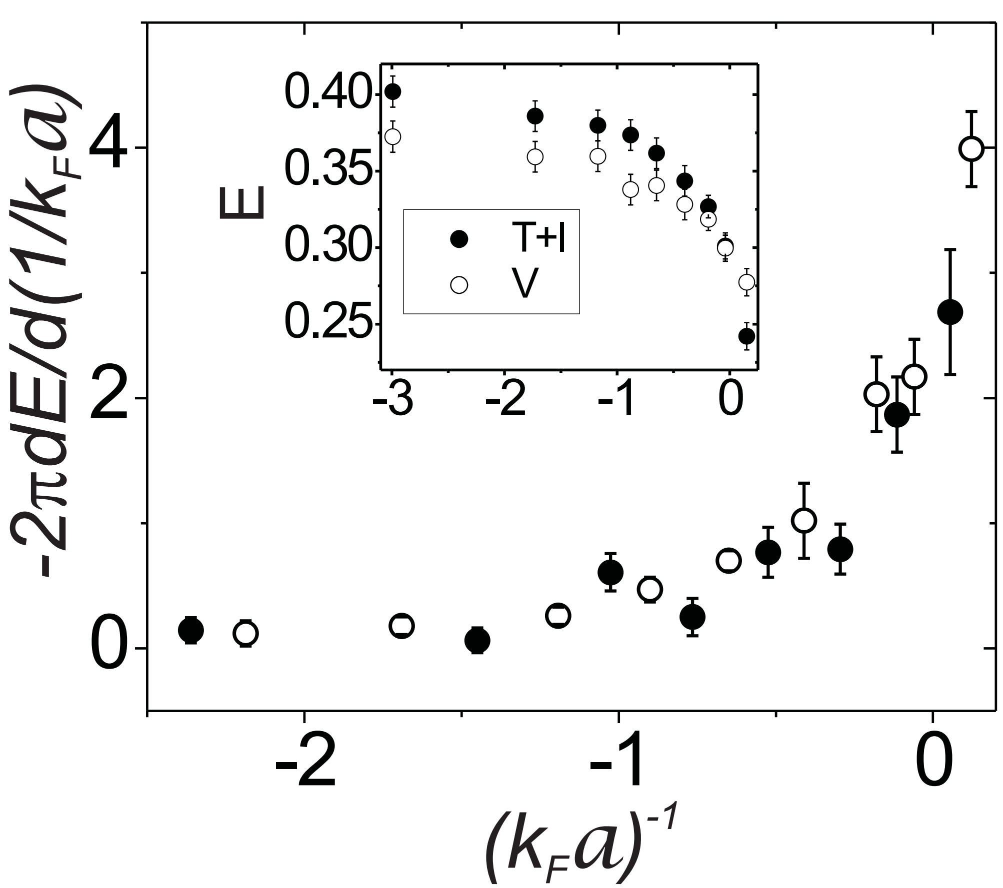
Experimental verification of Tan's relations

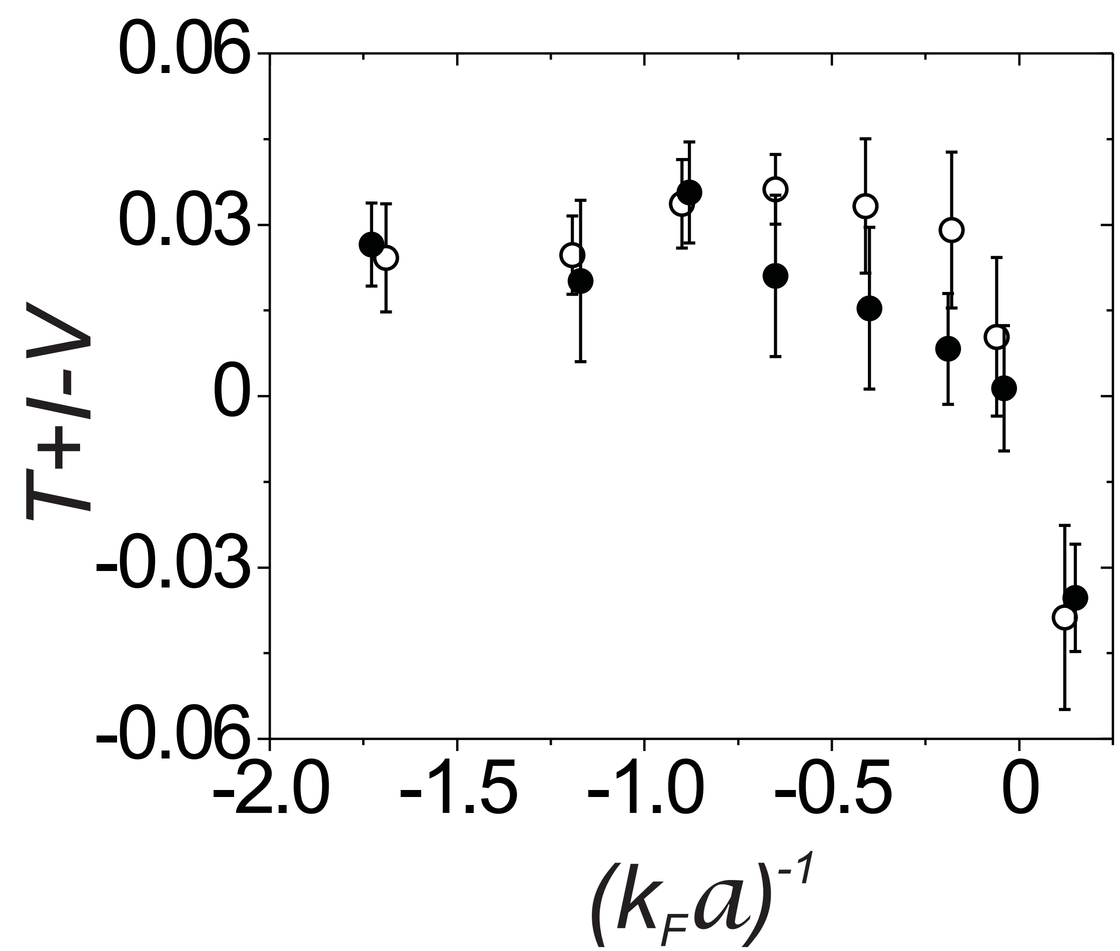
Contact measured directly from momentum distribution of the density (harmonically trapped ^{40}K atoms)

J.T. Stewart, *et al.*, Phys. Rev. Lett. **104** (2010) 235301

$$\boxed{C} = \boxed{(-4\pi M)\frac{\partial E}{\partial a^{-1}}} \qquad \boxed{\langle H\rangle = \langle V\rangle}\boxed{-\frac{C}{8\pi M a}}$$

filled
open





Strategies

- Experimental measurement (ultracold atoms)
  - enormous advances in techniques and precision
  - primary driving force behind theoretical investigations
- Analytical calculations
  - mean field, extrapolation from BCS regime, ε-expansion,...
  - generically possess uncontrolled systematic errors
- Numerical calculations
  - **Quantum Monte Carlo**
  - Density Functional Theory
  - ...

Numerical strategies (too many!)

- **D**iffusion **M**onte **C**arlo variational
  - coordinate space (continuum), fixed-node wavefunctions
- **G**reens **F**unction **M**onte **C**arlo variational
- **A**uxiliary **F**ield **M**onte **C**arlo
  - interactions induced via auxiliary fields
  - most similar to lattice QCD methods
- (**B**old-line) **D**iagrammatic **M**onte **C**arlo

  N. Prokof'ev, B. Svistunov
  Phys. Rev. Lett. **99** (2007) 250201
  - samples the space of Feynman diagrams
  - simulation performed in the thermodynamic limit
- **D**iagrammatic **D**eterminant **M**onte **C**arlo

Numerical strategies (here's one)

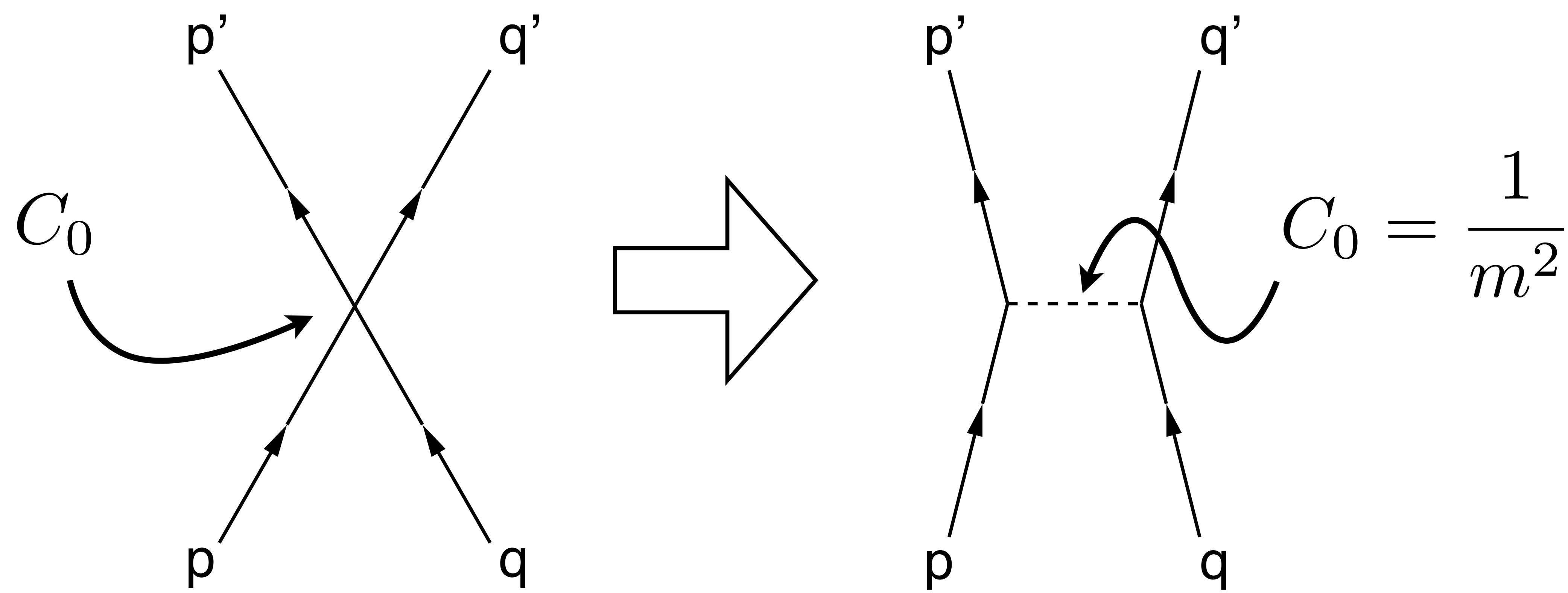
- **A**uxiliary **F**ield **M**onte **C**arlo
  - interactions induced via auxiliary fields
  - most similar to lattice QCD methods

Nice review article:
"Lattice methods for strongly interacting many-body systems"
J. E. Drut and A. N. Nicholson, J. Phys. G **40** (2013) 043101

Auxiliary Field Monte Carlo

$$S = \int d^{D+1}x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu + \phi \right) \psi + \frac{m^2}{2} \phi^2 \right]$$

Hubbard-Stratonovich transformation



Auxiliary Field Monte Carlo

$$Z = \int [d\phi] e^{-\frac{m^2}{2}\phi^2} \det\left(\partial_\tau - \frac{\nabla^2}{2M} - \mu + \phi\right)^2$$

(positive definite)

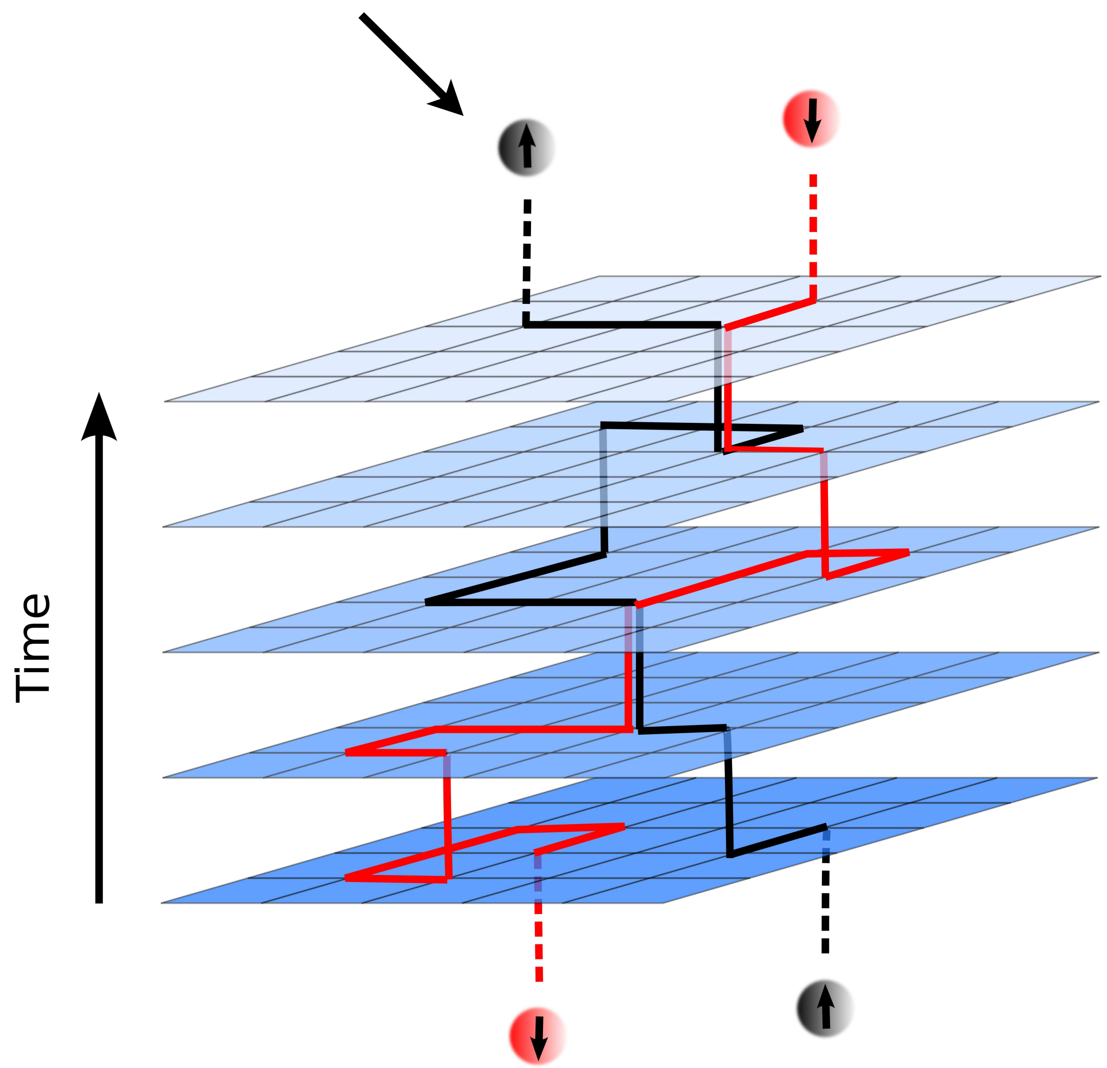
zero-temp.
(open BCs)

$$\det\left(1 - \frac{\nabla^2}{2M}\right)^{2N_\tau}$$

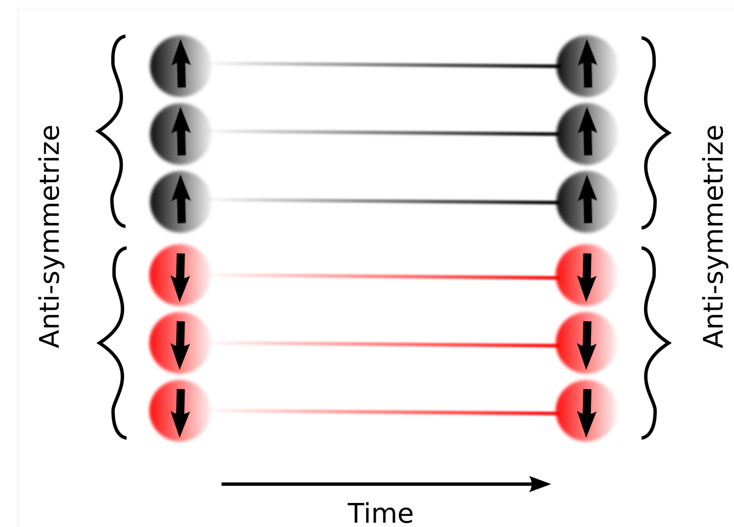
Single fermion propagator: $S(\phi) = \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu + \phi\right)^{-1}$

Fermions propagating on a random background

$$\psi_j'(\tau)^\dagger S(\phi)\psi_i(0)$$



$$S(\phi) = \left(\partial_\tau - \frac{\nabla^2}{2M} - \mu + \phi\right)^{-1}$$

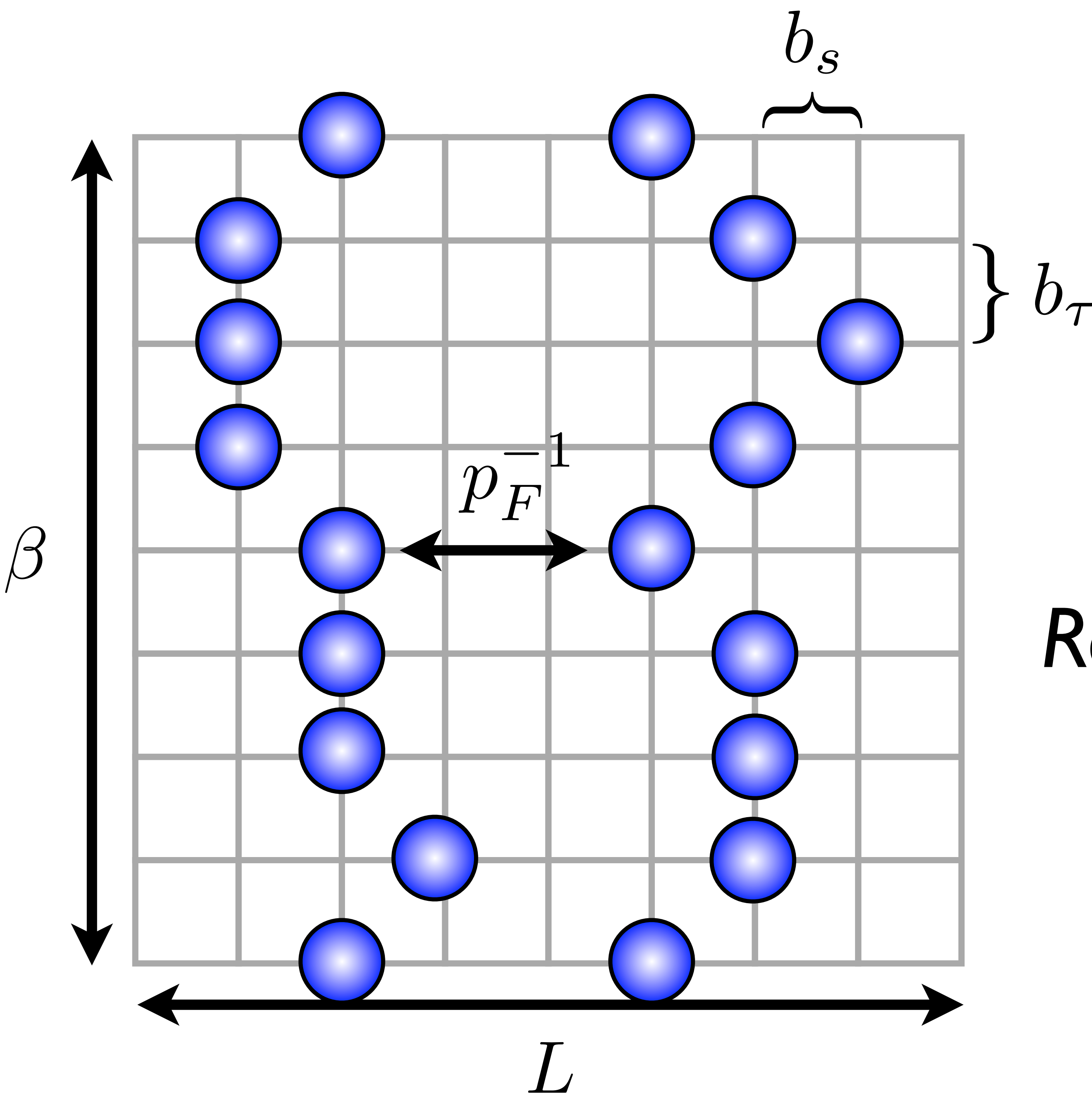
$$C_N(\tau) = \Bigg\langle$$


$$\Bigg\rangle_\phi$$

$$= \langle \Psi' | e^{-b\tau H} \cdots e^{-b\tau H} | \Psi \rangle$$

$$\sim e^{-b_\tau N_\tau E_{gnd}} + \text{excited}$$

Numerical strategies: Lattice Monte Carlo



Required hierarchy of scales:

$$r_0 \sim b_s \ll p_F^{-1} \ll L \ll a$$

$$b_\tau \ll E_F^{-1} \ll \beta$$

$$1 \ll N$$

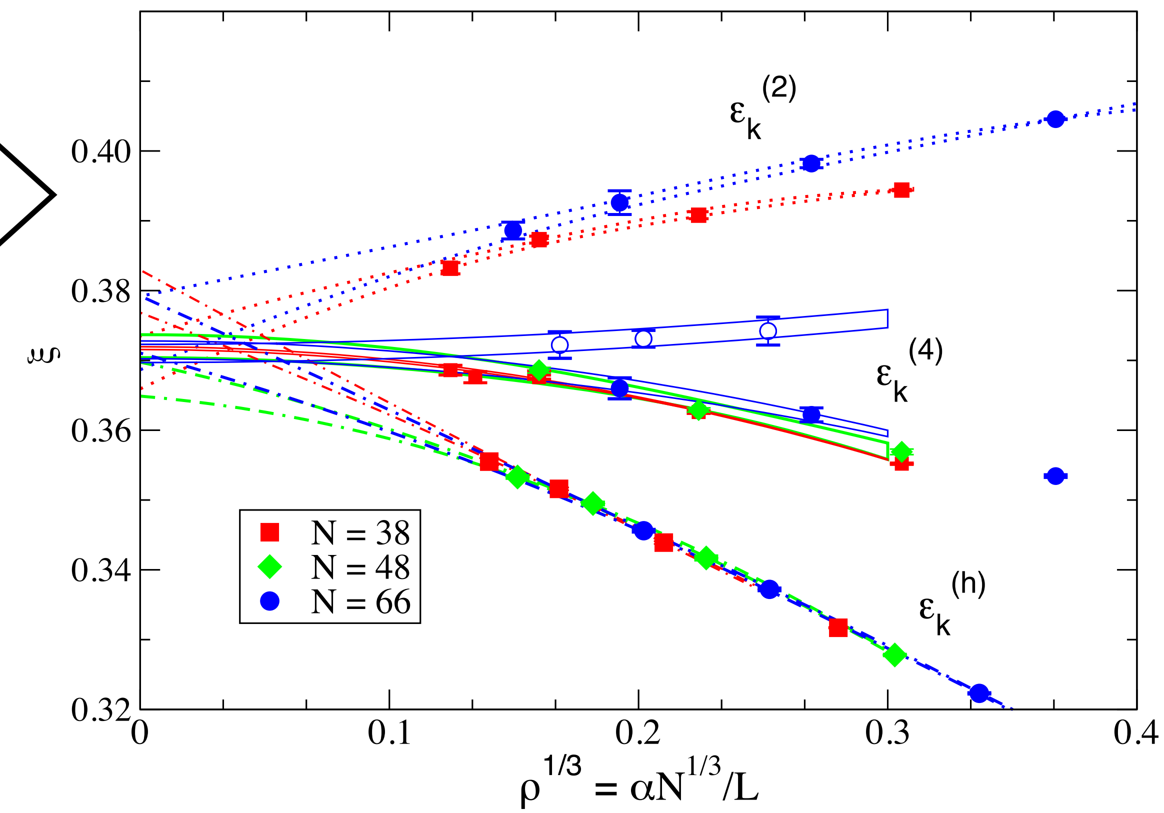
Removal of systematic errors

For <u>canonical</u> ensemble simulations, continuum limit is the same as the infinite volume limit!

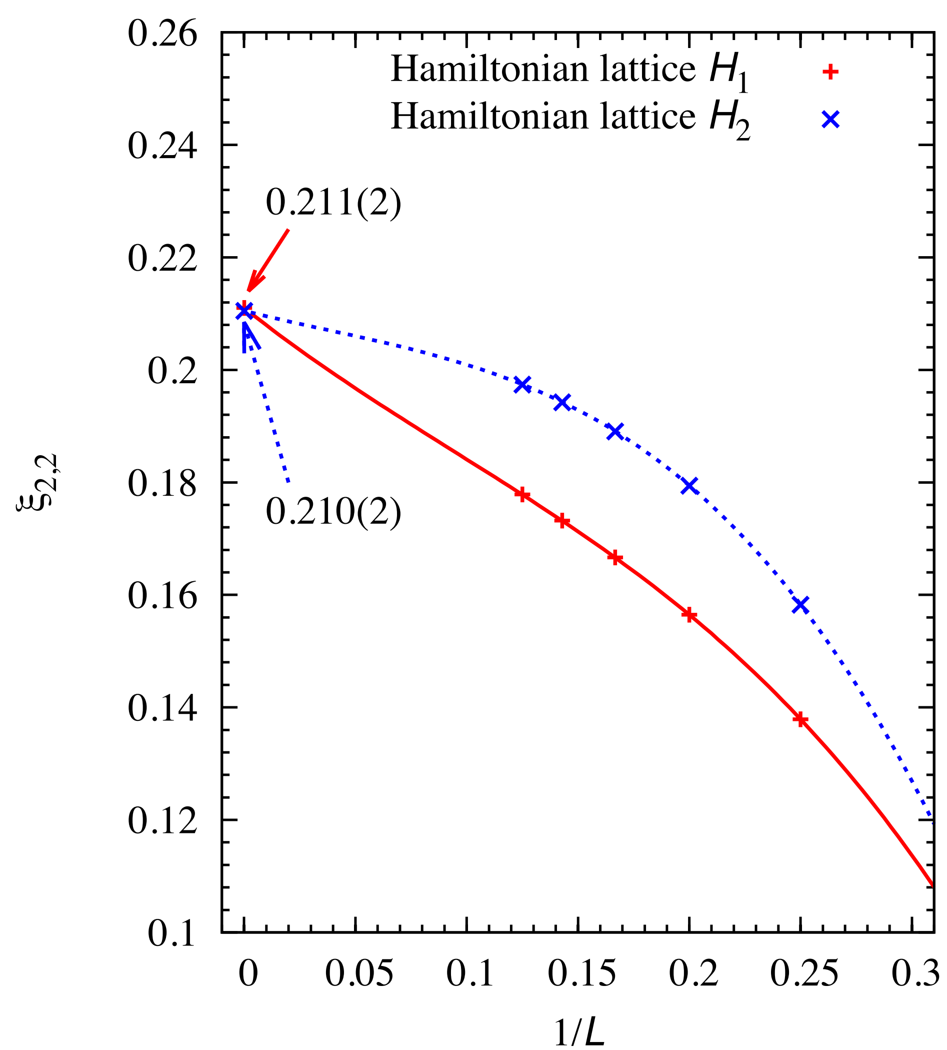
- Tune the scattering length to infinity
- Estimate observables, measured in appropriate units of L (L_0) and E_F (ω)
- Perform infinite volume extrapolations
- Repeat for several values of N and extrapolate N to infinity

Removal of effective range effects

...via modification of the single particle dispersion relation



J. Carlson, *et al.* Phys. Rev. A **84** (2011) 061602

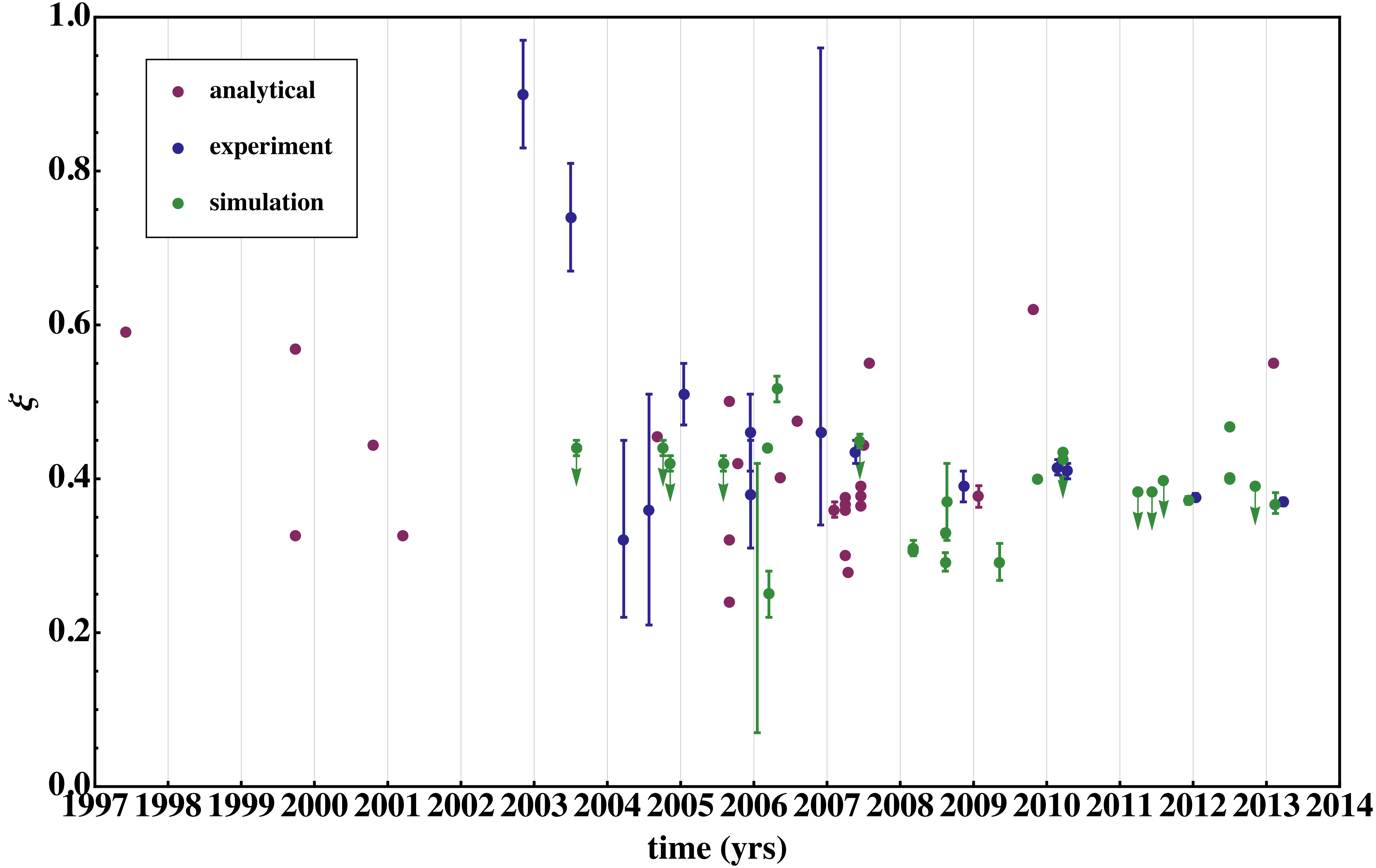


S. Bour, *et al.*, Phys. Rev. A ***83** (2011) 063619*

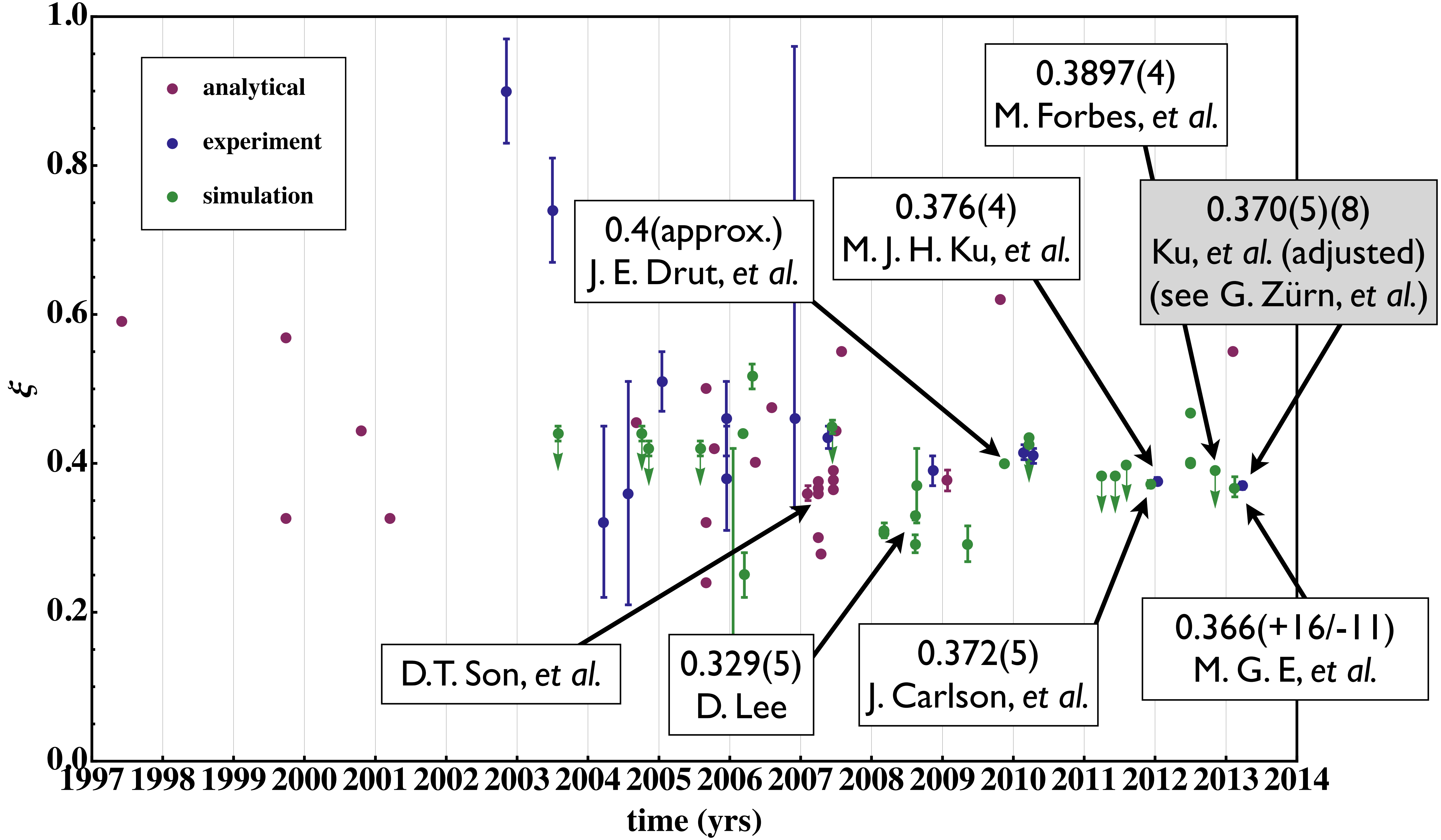
...via smearing the two-body interaction (i.e., adding derivative interactions)

See e. g.,
S. Bour, *et al.*, Phys. Rev. A ***83** (2011) 063619*
M. G. E, *et al.*, Phys Rev. A **84** (2011) 043644
J. Drut, Phys. Rev. A ***86** (2012) 013604*

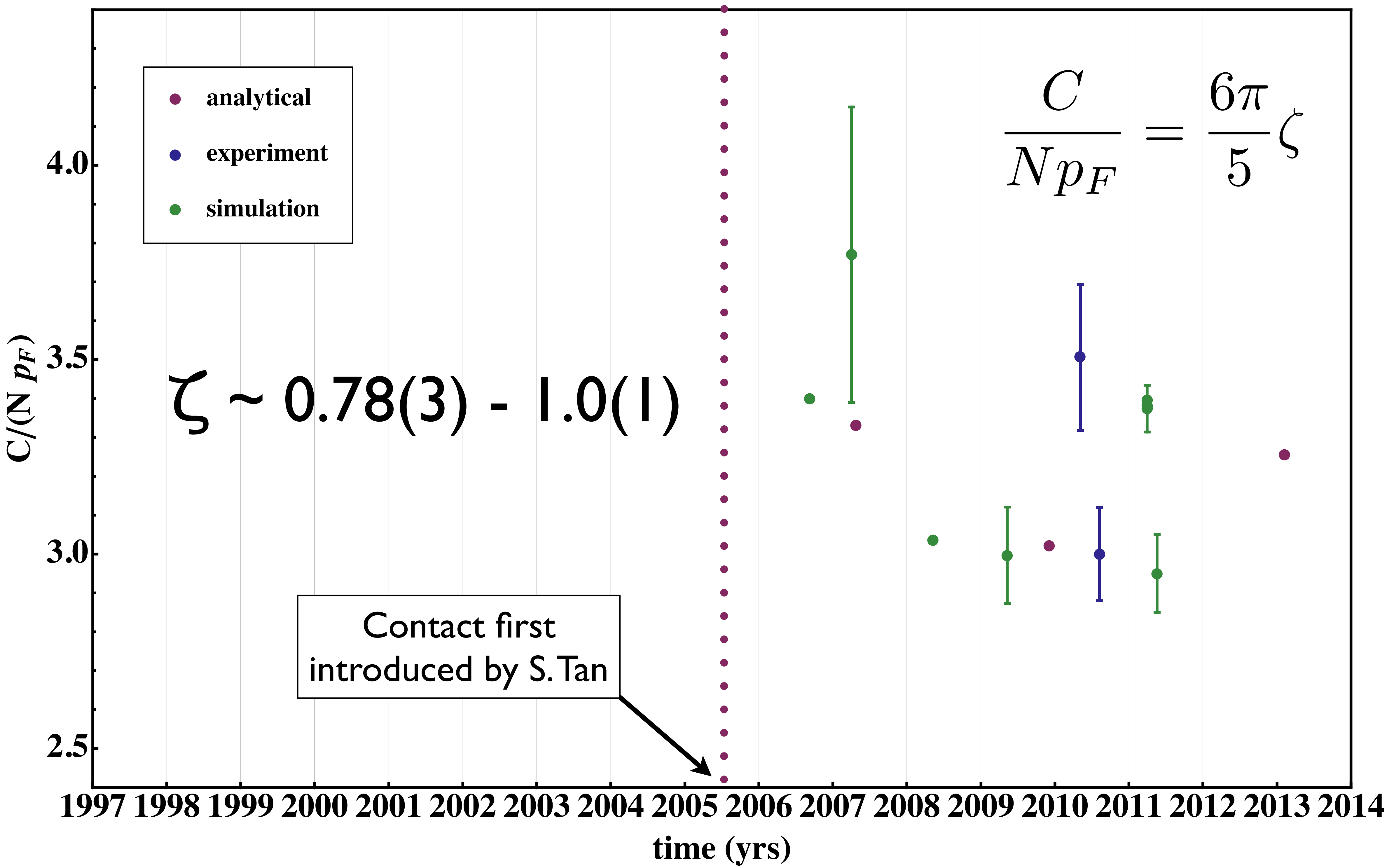
Bertsch parameter (ξ)



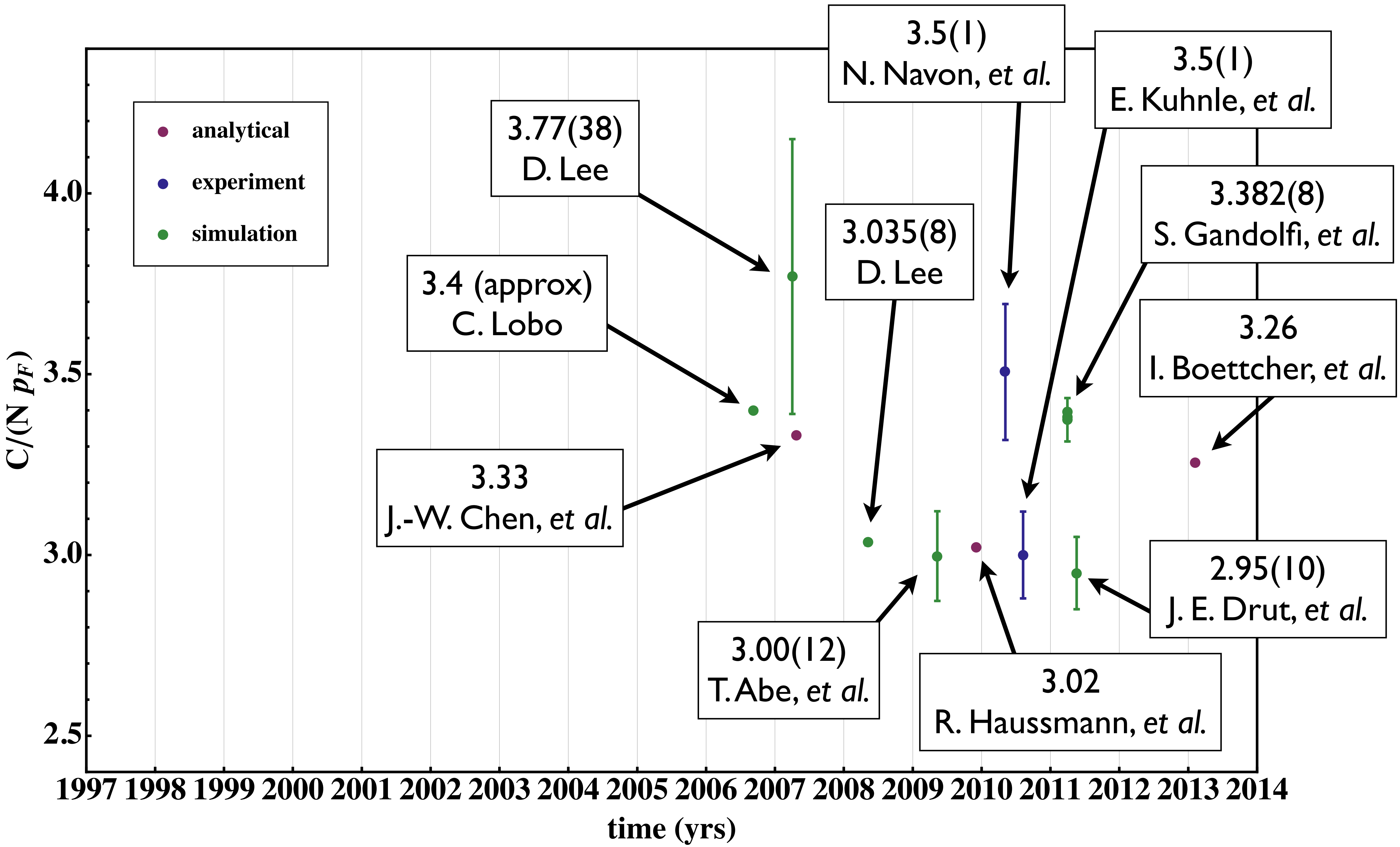
Bertsch parameter (ξ)



Dimensionless contact (homogeneous)



Dimensionless contact



Pairing gap (Δ)

$$\Delta(2n+1) = E(2n+1) - \frac{1}{2}\left[E(2n) + E(2n+2)\right]$$

$$N = 2n+1 \ (n \in \mathbb{Z})$$

J. Carlson, *et al.* (2003)

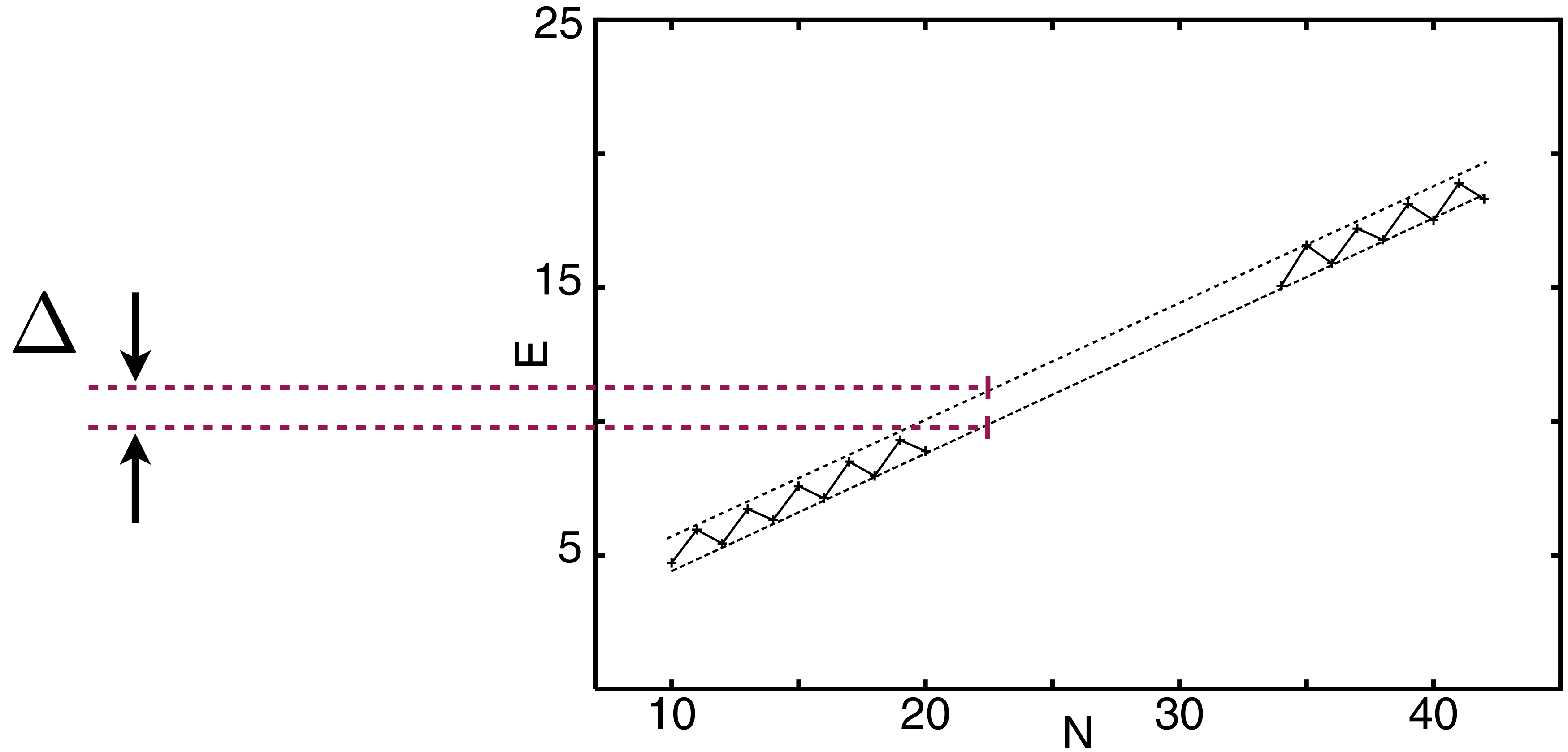
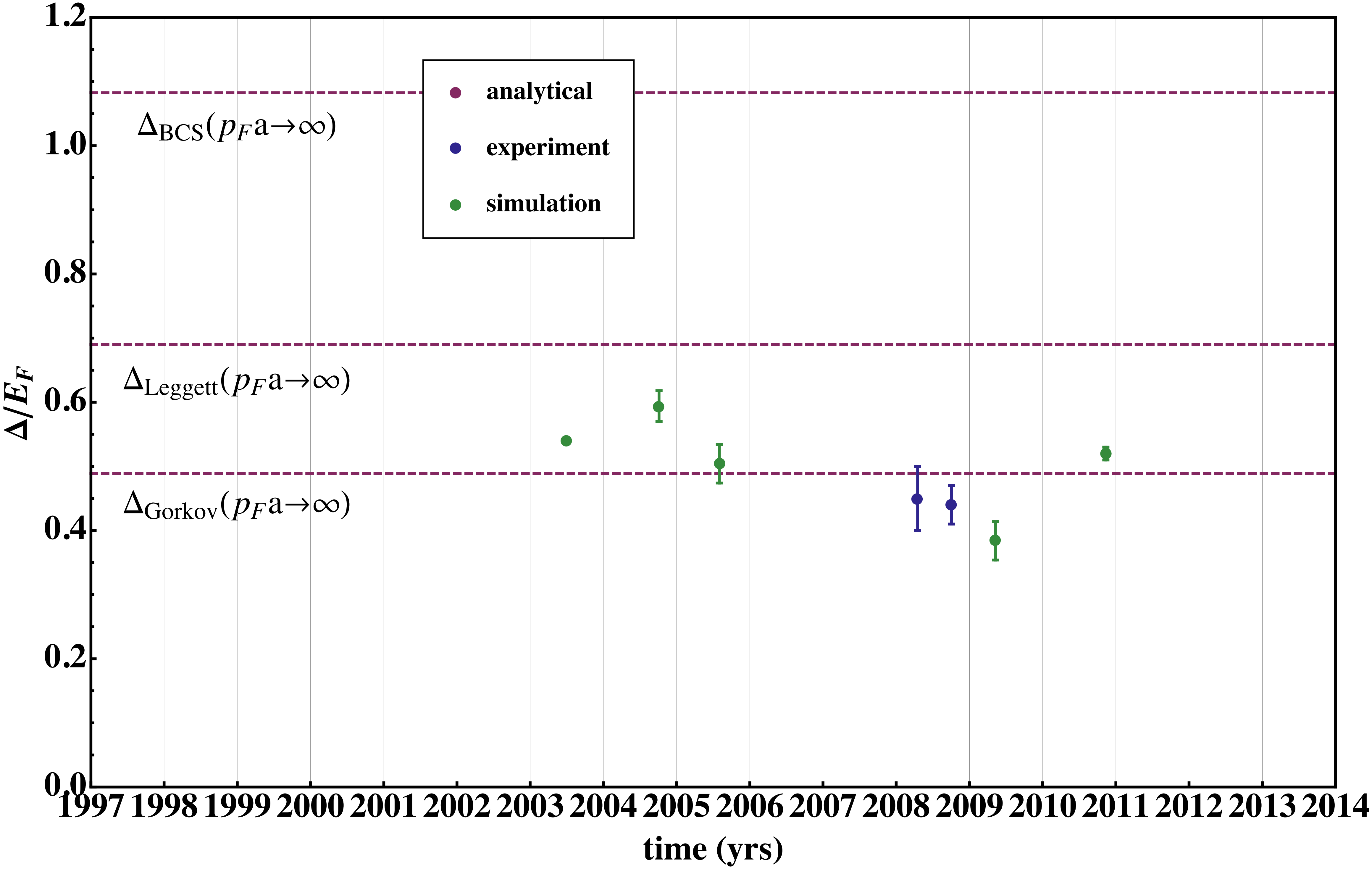
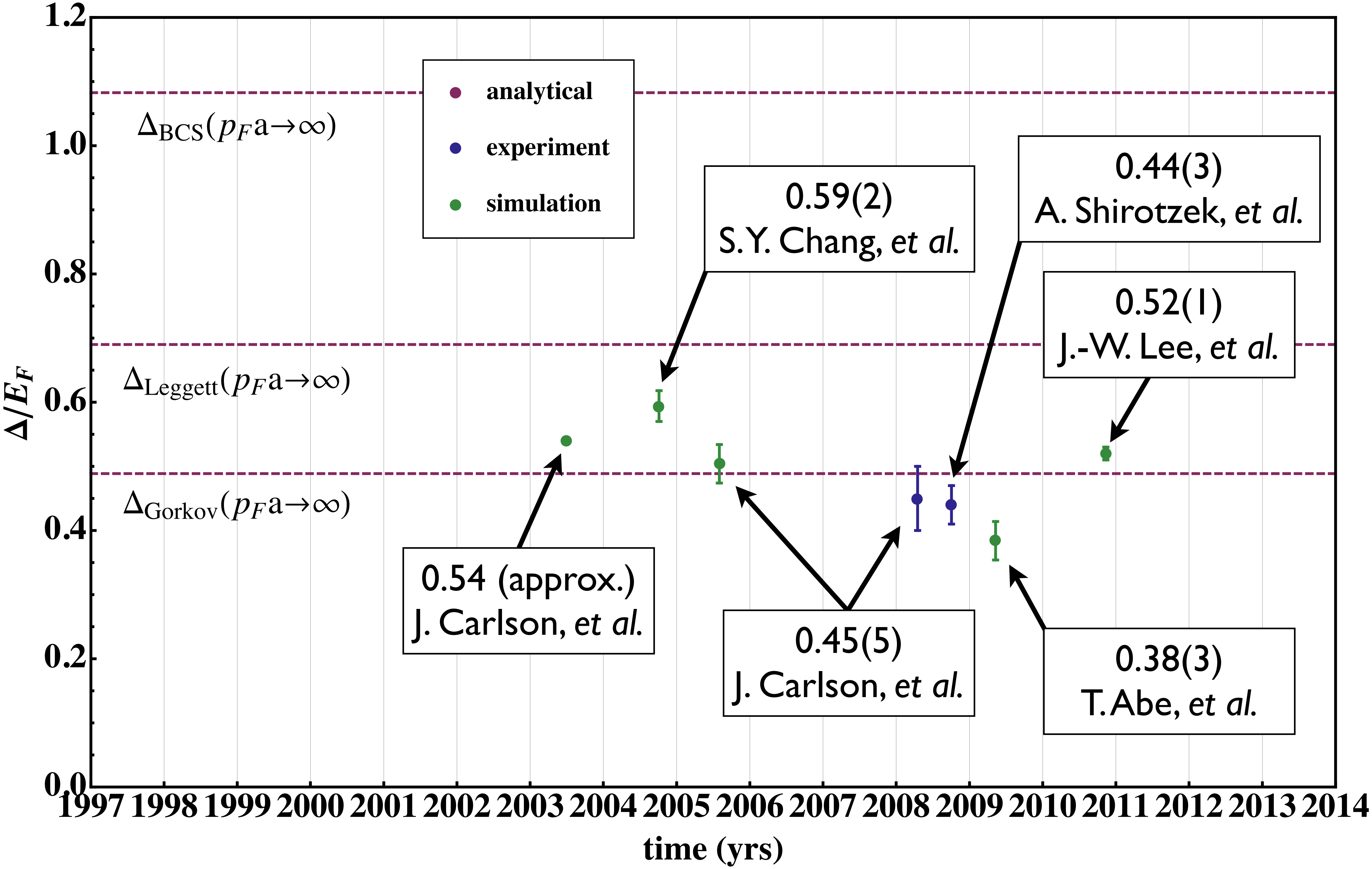


FIG. 3. The $E(N)$ in units of E_{FG}.

Pairing gap (Δ)



Pairing gap (Δ)



Finite Temperature

- Superfluid-normal phase transition temperature $T_c/E_F \sim O(1)$
  - experimentally and numerically accessible
  - $T_c/E_F \sim 0.167(13)$ (exp.) M. J. Ku, *et al.*, Science **335** (2012) 563
  - $T_c/E_F \sim 0.171(5)$ (DDMC) O. Goulko, *et al.*, Phys. Rev. A **82** (2010) 053621; O. Goulko, Lattice 2013 (poster session)
  - **many many** numerical studies
- Shear viscosity
  - evidence η/s approaches Kovtun-Son-Starinet (KSS) bound slightly above T_c
  - $\eta/s \sim 0.2\ \hbar/k_B$ G. Wlazłowski, *et al.*, Phys. Rev. Lett. **109** (2012) 020406

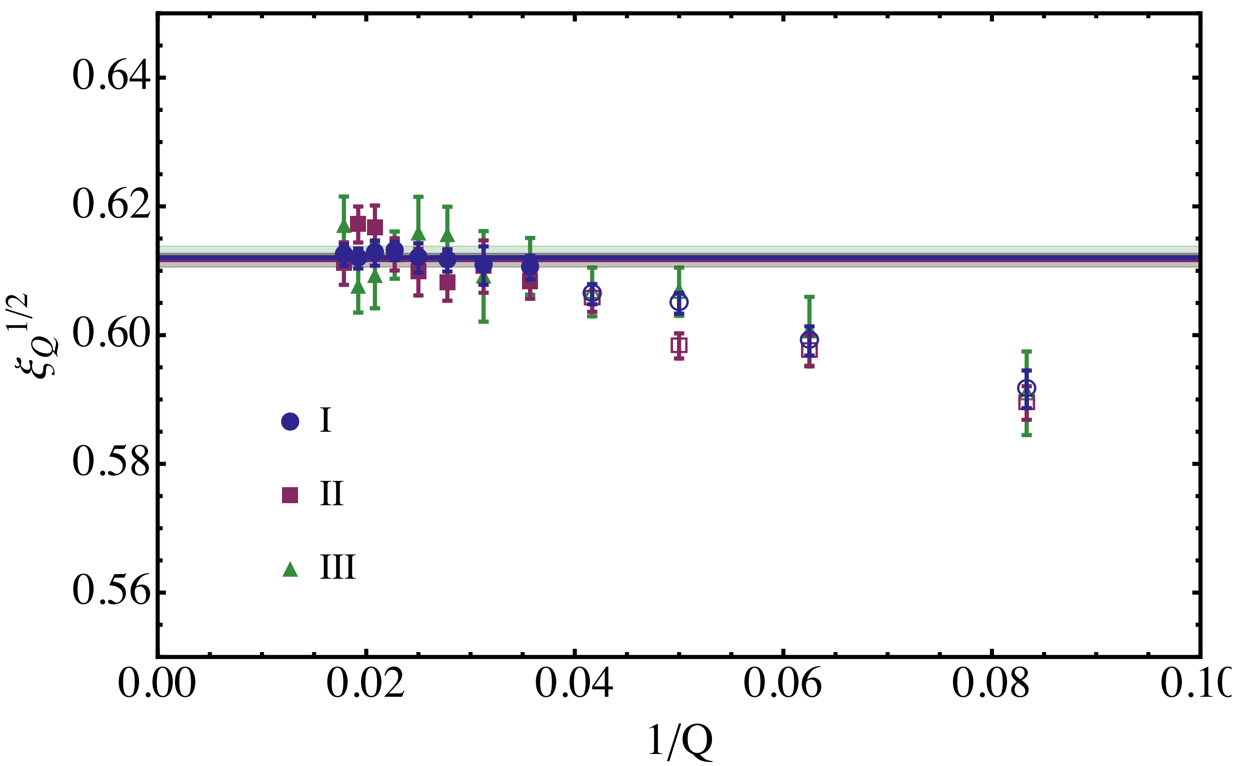
Universal fermi gases in lower dimensions

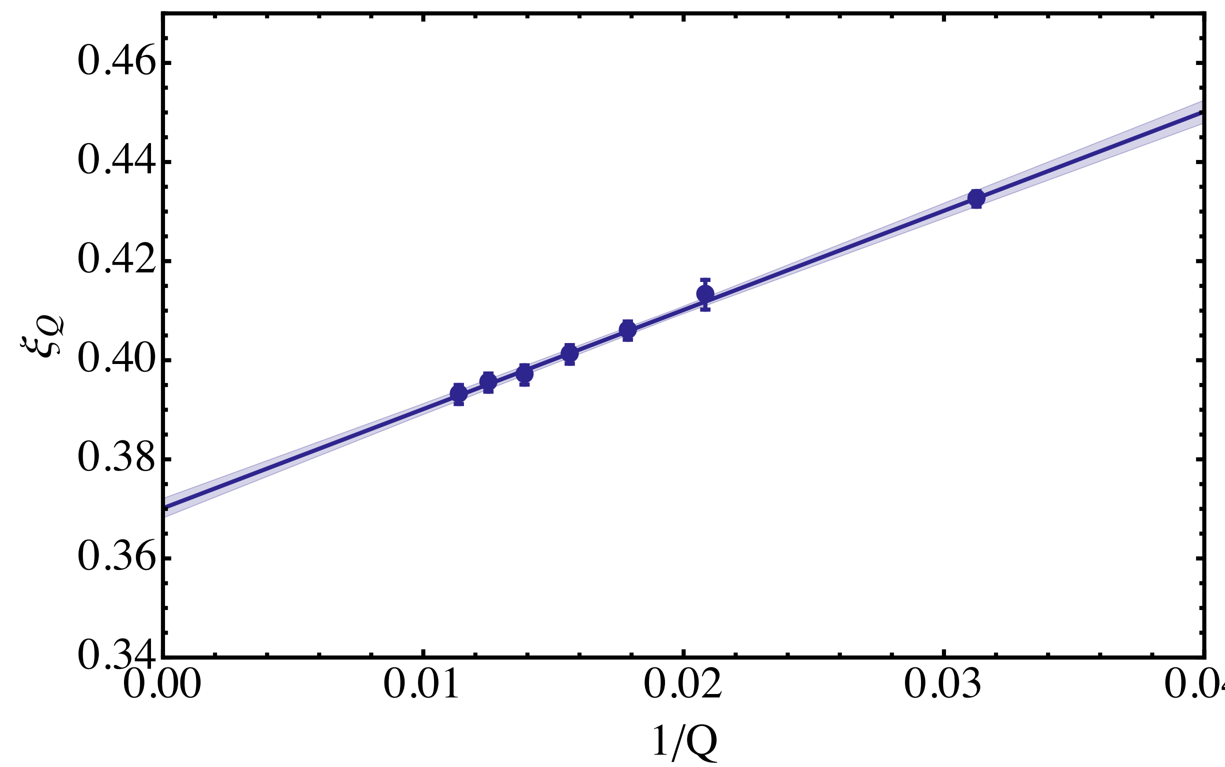
"Universal Fermi gases in mixed dimensions"
Y. Nishida and S. Tan
Phys. Rev. Lett. **101**, 170401 (2008)

"Universal four-component Fermi gas in one dimension"
Y. Nishida and D. T. Son
Phys. Rev. A **82**, 043606 (2010)

- 1+1 space-time dimensions
- 4-component Fermi gas
- Attractive 4-body contact interaction tuned to zero-energy bound state
- Qualitatively identical zero-temperature properties as in 3D

Unitary Fermions in one-dimension





$$\lim_{Q\to\infty} \xi_Q = 0.375(1)$$

$$\lim_{Q\to\infty} \xi_Q = 0.374(1)$$

$$\lim_{Q\to\infty} \xi_Q = 0.375(2)$$

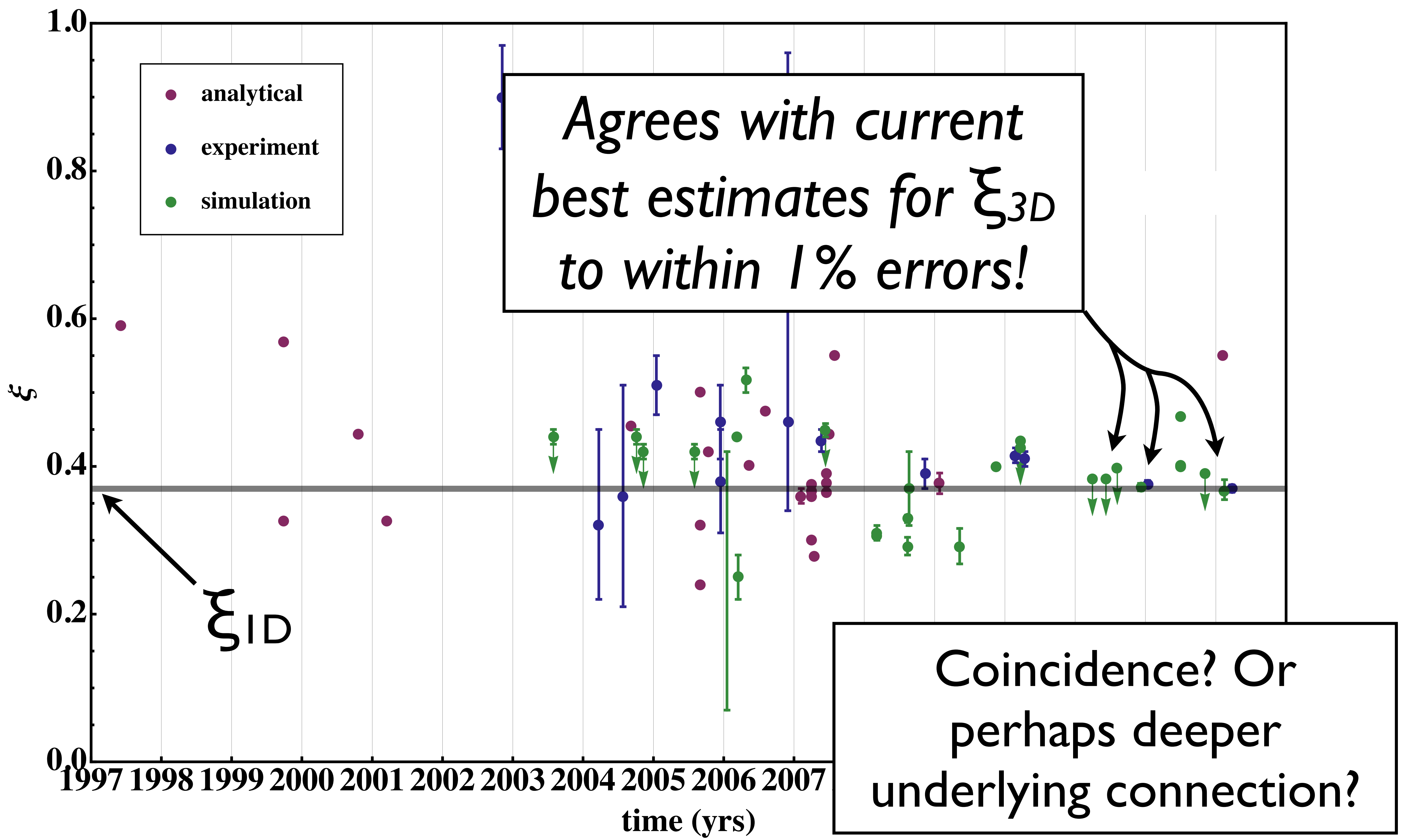
$$\lim_{Q\to\infty} \xi_Q = 0.370(2)$$

Also by combining contacts: ξ*=0.372(8)*

M. G. E., Phys. Rev. Lett. **109** (2012) 250403

M. G. E., Phys. Rev. A **87** (2013) 063617

Comparison of 1d and 3d Bertsch parameters



Summary

- Unitary fermions are an exciting subject!
  - strong interplay between experiment and theory
- Despite their apparent simplicity, unitary fermions remain challenging to study nonperturbatively
  - substantial progress in analytical and numerical techniques
  - provide a testing ground for new theoretical methods
  - widely varied results from simulation (e.g. contact)
  - many remaining open questions, new avenues for exploration
- Potential connections between unitary fermions in different dimensions?

Thank you for your attention!